Chapter 6: Work and Kinetic Energy

The Dot Product (Scalar Product)

For any two vectors $\vec{A} = \langle A_x, A_y, A_z \rangle$ and $\vec{B} = \langle B_x, B_y, B_z \rangle$, the dot product, or scalar product, is defined to be:

$$\vec{A} \cdot \vec{B} \equiv A_x B_x + A_y B_y + A_z B_z$$  \hspace{1cm} (1)

This is equivalent to:

$$\vec{A} \cdot \vec{B} \equiv \vec{A} \parallel \vec{B} \cos \phi,$$  \hspace{1cm} (2)

in which $\phi$ is the angle between $\vec{A}$ and $\vec{B}$.

- Note that $\vec{A} \cdot \vec{B}$ is just a number (scalar), not a vector! (Hence, the name “scalar product”.)
**Work**

When a force acts on an object *while the object undergoes a displacement*, the force can do some *work* on the object.

**Work Done By a Constant Force**

*Straight-Line Paths in 2-D and 3-D*

If the force is *constant* (in magnitude and direction), then the work done is given by:

\[
W = \vec{F} \cdot \Delta \vec{r} 
\]

\[
W = F \Delta r \cos \phi ,
\]

in which \( \phi \) is the angle between \( \vec{F} \) and \( \Delta \vec{r} \).

For straight-line paths in 3-D, we could have \( \vec{F} = \langle F_x, F_y, F_z \rangle \) and \( \Delta \vec{r} = \langle \Delta x, \Delta y, \Delta z \rangle \). (Both the force and displacement could have three non-zero components.) Then:

\[
W = F_x \Delta x + F_y \Delta y + F_z \Delta z
\]

For straight-line paths in 2-D (the x-y plane, e.g.), this simplifies to:

\[
W = F_x \Delta x + F_y \Delta y
\]
Paths in 1-D
For a 1-D path (along the x axis, say), \( \Delta y = \Delta z = 0 \), so (5) simplifies to

\[
W = F_x \Delta x
\]

(6)

and (4) becomes

\[
W = F \Delta x \cos \phi
\]

(7)
Notes:

• Unit (SI): \( N \cdot m \equiv "Joule", J \) (in honor of James Joule)

• \( W \) is a scalar, not a vector!
  o If there is more than one force doing work, we get the net work done by just adding the individual amounts of work like numbers!
    ▪ For \( N \) forces doing work:
      \[ W_{\text{net}} = W_1 + W_2 + \cdots + W_N \] (8)

• \( W \) can be positive, negative, or 0:
  o In (4), \( F \) and \( \Delta r \) are just the magnitudes of \( \vec{F} \) and \( \Delta \vec{r} \), so these are intrinsically positive. So the sign (+/-) of \( W \) is the same as the sign (+/-) of \( \cos \phi \):
    ▪ If \( \cos \phi \) +, \( W \) + ⇒ \(-90^\circ < \phi < +90^\circ\)
    ▪ If \( \cos \phi \) -, \( W \) - ⇒ \(+90^\circ < \phi < +180^\circ \) or \(-180^\circ < \phi < -90^\circ\)
  o If \( \cos \phi = 0 \), \( W = 0 \) ⇒ \( \phi = \pm90^\circ \)

(i.e., when the force and displacement are \( \perp \))
The Work-Energy Theorem

- Connection between the work done by the net force acting on an object and the change in the object’s kinetic energy.

\[ W_{\text{net}} = \Delta K \]  \hspace{1cm} (9)

\( W_{\text{net}} \) is the net work done: work done by the net force (all forces taken together).

\( K \) is the object’s kinetic energy.
To see that Eq. (9) holds for the case of an object moving in 1-D under the action of a constant net force, consider the work done:

\[ W_{net} = F_{net}\Delta x \]

By Newton’s 2\textsuperscript{nd} law, this becomes:

\[ W_{net} = ma_x\Delta x \quad (*) \]

And if the net force is assumed to be constant, then \( a_x \) is also constant, so we can use the 1-D constant acceleration equations. In particular,

\[ v_f^2 = v_i^2 + 2a_x\Delta x \]

Rearranging this for \( a_x \) and plugging this into (*) gives

\[ W_{net} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \quad (**), \]

If we now define \( \frac{1}{2}mv^2 \) to be the kinetic energy

\[ K \equiv \frac{1}{2}mv^2 \quad (10) \]

Then (**), becomes

\[ W_{net} = \Delta K, \]

Which is the work-energy theorem.
For the more general case of a non-constant force and a path that isn’t a 1-D path, the work-energy theorem still holds, but we have to do an integral to get the work. More about this a little bit later.
Notes on kinetic energy, $K$:

- Kinetic energy is energy that **moving** objects have because of their motion.
- SI unit: $\text{kg} \cdot \left(\frac{\text{m}}{\text{s}}\right)^2 = \text{N} \cdot \text{m} = \text{J}$
- $K$ is a **scalar**.
- $K$ is never negative.
- Only **moving** objects have kinetic energy; if $v = 0$, $K = 0$. 


Work and Energy with Varying Forces

1-D path
We said, for a constant force,\[ W = F_x \Delta x \]
(See Equation (6).)

But what if \( F_x \) varies as \( x \) changes? (That is, what if \( F_x \) is a function of \( x \)?)

In this case, the work becomes an integral. We will come back to this point after we discuss integrals.
Antiderivatives and the Indefinite and Definite Integrals

Antiderivatives
Let $F(x)$ be some function of $x$. An antiderivative of $F(x)$ is a function $f(x)$ whose derivative is $F(x)$:

$$F(x) = \frac{d}{dx}[f(x)]$$

Finding the Antiderivative
At this point in the course, we will discuss only polynomial functions of $x$. Let:

$$F(x) = a_n x^n$$  \hspace{1cm} (11)

Then:

$$f(x) = \frac{a_n}{n+1} x^{n+1}$$  \hspace{1cm} (12)

Check:

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}\left[\frac{a_n}{n+1} x^{n+1}\right] = \frac{a_n (n+1)}{n+1} x^{(n+1)-1} = a_n x^n = F(x)$$
Rule for Finding Antiderivatives of Polynomials
- increase the exponent by 1
- divide by the new exponent

In the most general case of a polynomial,
\[ F(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n, \quad (13) \]
the antiderivative is found by applying this rule term by term:
\[ f(x) = a_0x^1 + \frac{a_1x^2}{2} + \frac{a_2x^3}{3} + \frac{a_3x^4}{4} + \cdots + \frac{a_nx^{n+1}}{n+1} \quad (14) \]
(Check.)
If $f(x)$ is an antiderivative of $F(x)$, then
$$g(x) = f(x) + C,$$
where $C$ is any arbitrary constant, is also an antiderivative of $F(x)$.

**Proof:**

$$\frac{d}{dx}[g(x)] = \frac{d}{dx}[f(x) + C] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[C] = \frac{d}{dx}[f(x)] + 0$$

But if $f(x)$ is an antiderivative of $F(x)$, then

$$\frac{d}{dx}[f(x)] = F(x),$$

so:

$$\frac{d}{dx}[g(x)] = F(x),$$

which says that $g(x)$ is also an antiderivative of $F(x)$. 
The Indefinite Integral

The antiderivative of $F(x)$ is also called the indefinite integral of $F(x)$, and is written

$$\int F(x) \, dx$$

So if $f(x)$ is an antiderivative of $F(x)$, then

$$\int F(x) \, dx = f(x) + C,$$  \hspace{1cm} (15)

in which $C$ is an arbitrary constant (independent of $x$).
The Definite Integral
Consider a graph of any arbitrary function $F(x)$. Suppose we wanted to find the area under the curve from $x = a$ to $x = b$. We can find an approximation to this area by dividing the area up into $N$ partitions, each of which is a rectangle. Let the width of each partition be $\Delta x$. The height of the $i$-th rectangle is then $F(x_i)$, the value of the function $F$ at $x = x_i$.

The area under the curve is then approximately:

$$A \approx F(x_1)\Delta x + F(x_2)\Delta x + \cdots + F(x_N)\Delta x$$

$$A \approx \sum_{i=1}^{N} F(x_i)\Delta x$$

This area is only approximately correct because the function $F(x)$ is not really constant over the width of one partition. But this approximate area approaches the exact area if we let the number of partitions become arbitrarily large, with the consequence that the width of each partition must approach zero. The exact area is thus

$$A = \lim_{N \to \infty} \left[ \sum_{i=1}^{N} F(x_i)\Delta x \right]$$

(16)
The quantity on the right-hand side immediately above is defined to be the **definite integral** of \( F(x) \) between \( x = a \) and \( x = b \), and is written
\[
\int_{a}^{b} F(x) \, dx
\]

So
\[
A = \int_{a}^{b} F(x) \, dx \quad (17)
\]

The integral sign , \( \int \), resembles an elongated letter “s”. It stands for “sum”, or, more correctly, the *limit* of the sum of all the terms \( F(x_i) \Delta x \) in (16). You can think of the definite integral \( \int_{a}^{b} F(x) \, dx \) as representing the sum of a bunch of infinitesimal terms \( F(x) \, dx \) from \( x = a \) to \( x = b \).
Evaluating the Definite Integral:
The Fundamental Theorem of Calculus

How do we evaluate the definite integral in (17) for some particular function $F(x)$? The Fundamental Theorem of Calculus answers this question.

*The Fundamental Theorem of Calculus*

The Fundamental Theorem of Calculus connects the definite integral with the concept of the antiderivative. This theorem (in one of its forms) says that if $f(x)$ is an antiderivative of $F(x)$, then

$$\int_{a}^{b} F(x) \, dx = f(b) - f(a) \quad (18)$$

Thus, the way you evaluate the definite integral of $F(x)$ is to take the antiderivative of $F(x)$ and evaluate the antiderivative at the *endpoints* of the interval $x = a$ to $x = b$. 
The Fundamental Theorem of Calculus can be understood with the help of the concept of the total differential of a function $f(x)$.

If $f$ is a function of $x$, then the total differential of $f$, written $df$, is defined by

$$df \equiv \frac{df}{dx} dx$$

(19)

This can be understood by considering any graph of $f(x)$ vs $x$. The quantity $dx$ represents some infinitesimal (i.e., arbitrarily small) change in $x$. If the interval $dx$ is arbitrarily small, then $f(x)$ can be approximated by the tangent line to $f(x)$. The quantity $(df/dx)dx$ is then the infinitesimal change in $f$ for any interval from $x$ to $x + dx$.

In (18), then, if $f(x)$ is an antiderivative of $F(x)$, then $F(x) = \frac{df}{dx}$, and the definite integral in (18) becomes:

$$\int_{a}^{b} F(x) dx = \int_{a}^{b} \frac{df}{dx} dx$$

(20)
or

\[
\int_{a}^{b} F(x) \, dx = \int_{f(a)}^{f(b)} df, \quad (21)
\]

using the definition of the total differential.

The integral \( \int_{f(a)}^{f(b)} df \) represents the sum of a bunch of infinitesimal changes in \( f \) from \( f(a) \) to \( f(b) \). This sum equals the total change in the function \( f \), \( \Delta f = f(b) - f(a) \). Thus (21) becomes

\[
\int_{a}^{b} F(x) \, dx = f(b) - f(a),
\]

which is the Fundamental Theorem of Calculus.
Work and Energy with Varying Forces (continued)

1-D path
Returning to our earlier discussion of calculating the work for a variable force, consider an object that moves in 1-D under the action of a variable force $F_x(x)$. The infinitesimal amount of work $dW$ done by this force as the object moves through an infinitesimal displacement $dx$ is $F_x(x)dx$. The total work done over any interval $x = a$ to $x = b$ is then

$$W = \int_{a}^{b} F_x(x) \, dx$$

(22)

The only kind of function $F_x(x)$ I will ask you to integrate at this point in the course is a polynomial function.
Curved Paths in 2-D and 3-D
Consider an object acted upon by a varying force \( \vec{F} \) as the object moves along an arbitrary path in 2-D or 3-D from a point \( P_i \) (with position vector \( \vec{r}_i \)) to a point \( P_f \) (with position vector \( \vec{r}_f \)). As the object moves through an infinitesimal displacement \( d\vec{r} \) along this path, the force \( \vec{F} \) does an infinitesimal amount of work \( dW \) given by

\[
dW = \vec{F} \cdot d\vec{r}
\]

The total work done as the object moves from \( \vec{r}_i \) to \( \vec{r}_f \) is then the sum of all these infinitesimal contributions:

\[
W = \int_{\vec{r}_i}^{\vec{r}_f} dW = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r} \tag{23}
\]

This is called the line integral of \( \vec{F} \) along the specified path from \( \vec{r}_i \) to \( \vec{r}_f \).
I won’t ask you to evaluate this integral in this course, but you will learn how to do this in Calculus III.
Power

The word “power” in physics means the rate at which energy is *supplied* or *dissipated* (“used up”):

\[ P \equiv \frac{dE}{dt} \quad (24) \]

*If* energy is supplied to a system by *work* being done *on* the system, then this becomes:

\[ P = \frac{dW}{dt} \quad (25) \]

For an arbitrary force and an arbitrary infinitesimal displacement,

\[ dW = \vec{F} \cdot d\vec{r} \]

This means we can write the power as:

\[ P = \frac{dW}{dt} = \frac{\vec{F} \cdot d\vec{r}}{dt} \]

\[ P = \vec{F} \cdot \vec{v} \quad (26) \]