

Chapter L

Interference, Diffraction and Polarization

Blinn College - Physics 2426 - Terry Honan

Simple Interference

A sinusoidal wave travelling in one dimension has the form:

$$A \cos(kx \mp \omega t)$$

where in the case of electromagnetic radiation the amplitude A is the peak electric field, $A = E_{\max}$. At some position x the disturbance varies with time by the general form

$$A \cos(\omega t + \phi).$$

This form generally describes waves that vary sinusoidally with time at some position even in two or three dimensions.

Now consider combining waves (at some position) from two sources with the same amplitude but with a different relative phase angle ϕ . To do this we just add the two functions of time

$$A_0 \cos \omega t + A_0 \cos(\omega t + \phi).$$

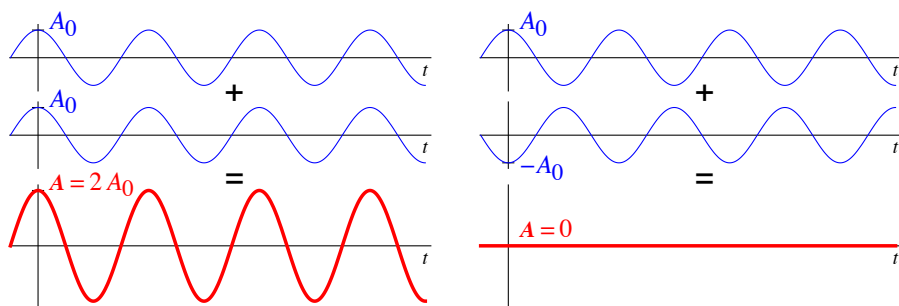
If the two waves are in phase $\phi = 0$ we get *constructive interference* and if they are totally out of phase $\phi = \pi$ we get *destructive interference*. In the constructive case

$$A_0 \cos \omega t + A_0 \cos \omega t = 2 A_0 \cos \omega t$$

and the resulting amplitude of the combined wave is $A = 2 A_0$. In the destructive case, since $\cos(\omega t + \pi) = -\cos \omega t$, we get.

$$A_0 \cos \omega t + A_0 \cos(\omega t + \pi) = 0.$$

with a resulting combined amplitude of $A = 0$.



Constructive Interference

Destructive Interference

The more general result is a bit more complicated.

$$A_0 \cos \omega t + A_0 \cos(\omega t + \phi) = A \cos\left(\omega t + \frac{\phi}{2}\right) \text{ where } A = 2 A_0 \cos \frac{\phi}{2}.$$

Before proving this result it should be mentioned that the special cases of constructive and destructive interference follow trivially: $\phi = 0$ gives $A = 2 A_0$ and $\phi = \pi$ gives $A = 0$. To verify the result, first start with the identity

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta.$$

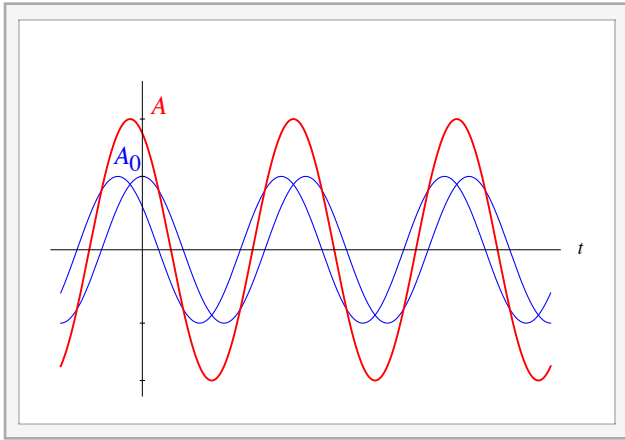
It follows that

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos \alpha \cos \beta.$$

Taking $\alpha = \omega t + \frac{\phi}{2}$ and $\beta = \frac{\phi}{2}$ gives

$$\cos(\omega t) + \cos(\omega t + \phi) = 2 \cos \frac{\phi}{2} \cos\left(\omega t + \frac{\phi}{2}\right)$$

which implies the desired result.



Interactive Figure - General Interference

The intensity is proportional to the square of the peak electric field and this is just the amplitude.

$$I = \frac{E_{\max}^2}{2 \mu_0 c} \implies I \propto A^2$$

If two flashlights, each with intensity I_0 shine at the same point the resulting intensity is $2 I_0$; this is incoherent mixing, meaning that the phase of light of one source is not related to the phase of the other. The key to interference is coherence; the phases of the two sources are related. The following table summarizes the amplitudes and intensities for the special cases of constructive and destructive interference, for the general case and for incoherent mixing. In all cases the amplitude and intensity of the uncombined waves are A_0 and I_0 .

	Amplitude of Combined Waves	Intensity of Combined Waves
Constructive Interference $\phi = 0$	$2 A_0$	$4 I_0$
Destructive Interference $\phi = \pi$	0	0
General Case (any ϕ)	$2 A_0 \cos \frac{\phi}{2}$	$4 I_0 \cos^2 \frac{\phi}{2}$
Incoherent Mixing (two flashlights)	–	$2 I_0$

Young's Double-slit Experiment

Consider monochromatic light of wavelength λ normally incident on a pair of narrow vertical slits each of width a and separated by d , the distance between the centers of the slits. We will make the simplifying assumption of narrow slits, or $a \ll d$. We will take L to be the distance from the slits to some distant screen; by distant we mean that L is much larger than both d and λ . A point P on the screen can be labelled by θ , the angle of deflection or by y , the distance along the screen, where $y = 0$ and $\theta = 0$ describe undeflected rays. θ and x are related by

$$\tan \theta = \frac{y}{L}.$$

The distance from on slit to P is r_1 and r_2 is the distance from the other slit to P . Each r is much larger than d but their difference δ is on the same order. Using trig and the approximation that $d \ll r$ we get $\delta = d \sin \theta$

$$\delta = r_2 - r_1 = d \sin \theta.$$

Constructive and Destructive Interference

Huygen's principle states that every point on a wave front behaves as a separate point source. When a plane wave hits a pair of narrow slits then each slit represents a different source. The key point is that each source is in phase with the other; this coherence is the key to interference. If the two distances r_1 and r_2 are equal (so $\delta = 0$) then the light rays hitting P are in phase and there is constructive interference. Moreover, if the path difference δ is an integer number of wavelengths there is constructive interference. If the path difference is half a wavelength then the rays are out of phase and there is destructive interference.

Using that the path difference is $d \sin \theta$ and taking m to be any integer we can write the conditions for constructive and destructive interference.

$$d \sin \theta = m \lambda \quad (\text{constructive interference})$$

$$d \sin \theta = \left(m + \frac{1}{2}\right) \lambda \quad (\text{destructive interference})$$

where $m = 0, \pm 1, \pm 2, \dots$.

Intensity Formula

We can generalize the preceding results. We can relate the path difference δ to the phase difference ϕ . Every wavelength path difference corresponds to a 2π phase difference, so we can write

$$\frac{\delta}{\lambda} = \frac{\phi}{2\pi} \implies \phi = \frac{2\pi}{\lambda} \delta = \frac{2\pi}{\lambda} d \sin \theta$$

since the intensity for the sum two waves of equal amplitude out of phase by ϕ satisfies $I \propto \cos^2 \frac{\phi}{2}$ we get

$$I = I_{\max} \cos^2 \left(\frac{\pi d \sin \theta}{\lambda} \right),$$

where I_{\max} is defined as the peak intensity, which is the value at $\theta = 0$.

Constructive and destructive interference are simple special cases of this intensity formula. The peak intensity is when \cos^2 is one and that is when $\frac{\pi d \sin \theta}{\lambda} = m \pi$; this implies the constructive interference formula. Destructive interference is when $I = 0$ and that is when $\frac{\pi d \sin \theta}{\lambda} = (m + \frac{1}{2}) \pi$, implying the destructive condition.

Small Angle Formula

It is a common case that the angle θ is small. This is equivalent to y being small compared to L .

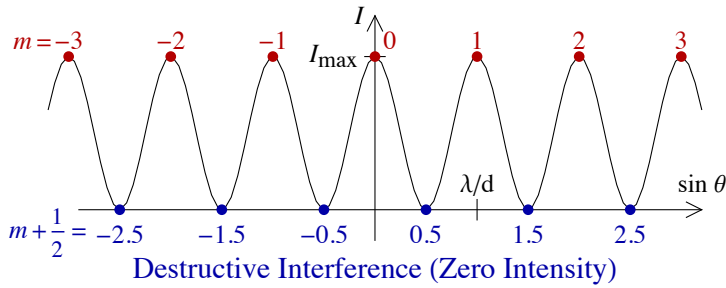
$$\theta \text{ is small} \iff y \ll L.$$

Since $\tan \theta = y/L$ and for small angles $\sin \theta \approx \tan \theta$ we can replace $\sin \theta$ with y/L ,

$$\sin \theta \approx \frac{y}{L} \text{ for small } \theta.$$

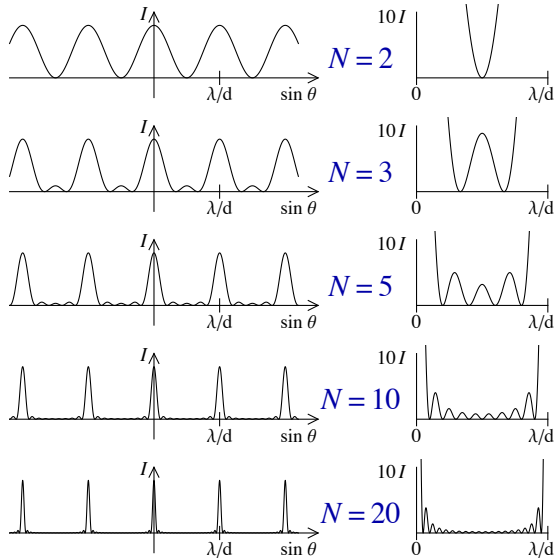
Graph of Intensity

Constructive Interference (Maximum Intensity)



Many Slits and Diffraction Gratings

Now consider the case of N narrow slits. The following graphs show the intensity graphs. The graphs on the right is an enlargement of one period of the graph where the intensity is multiplied by 10. Note that the condition for constructive interference is the same but as the number of slits increases the intensity goes to zero at all positions in between.



As an aside, it should be mentioned that the intensity formula for the above graphs is

$$I = I_{\max} \left[\frac{\sin(N k d \sin \theta)}{N \sin(k d \sin \theta)} \right]^2 \quad \text{where } k = \frac{2\pi}{\lambda}.$$

A diffraction grating is the limit as N becomes large. The interference conditions for a diffraction grating become

$$d \sin \theta = m \lambda \quad \text{for constructive interference}$$

and destructive interference elsewhere.

Note that the position of the m^{th} maximum varies with wavelength. If white light passes through a diffraction grating then the central fringe ($m = 0$) is the same for all wavelengths and thus is white, but the higher order fringes will break light into its spectrum. Diffraction gratings are better for spectroscopy (resolving light into its constituent wavelengths) than prisms.

Single-slit Diffraction

When light passes through a single narrow slit of width a one observes a diffraction pattern. Here each point in the slit is a separate source. It is simple to find the condition for destructive interference. If light from one point in the slit travels half a wavelength further than light from half the slit ($a/2$) away then $\delta = r_2 - r_1$ and

$$\delta = \frac{a}{2} \sin \theta = \frac{\lambda}{2}$$

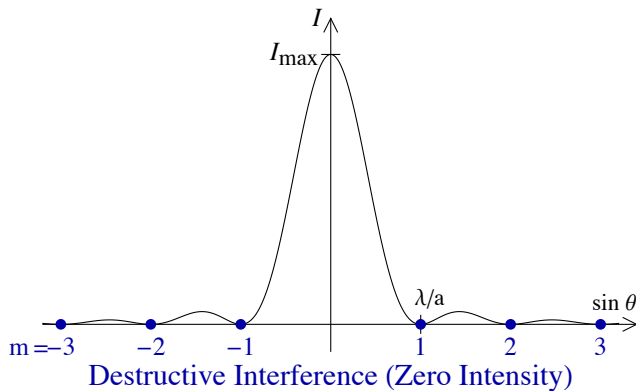
is a condition for destructive interference, since every ray is exactly cancelled by another. The same argument works if we split the slit into an even number of pieces $2m$ and light from $\frac{a}{2m}$ away and

$$\delta = \frac{a}{2m} \sin \theta = \frac{\lambda}{2}$$

Note that this argument doesn't work for odd numbers since there will always be an unpaired ray. The condition for destructive interference is then

$$a \sin \theta = m \lambda, \text{ where } m = \pm 1, \pm 2, \pm 3, \dots \text{ (destructive interference)}$$

Below is the graph of the intensity pattern for a single slit. Note that the $m = 0$ position is not destructive interference; it is the center of the central bright fringe.



As an aside, the intensity formula for the above graph is

$$I = I_{\max} \left[\frac{\sin[(\pi a/\lambda) \sin \theta]}{(\pi a/\lambda) \sin \theta} \right]^2.$$

Thin Film Interference and Phase Change under Reflection

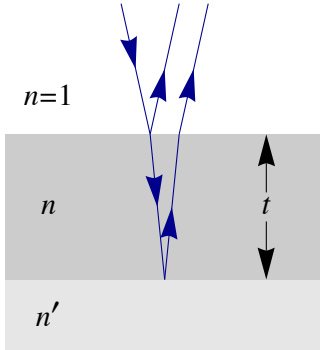
Phase Change under Reflection

When any wave moves from one medium to another there is a reflected wave and a refracted (or transmitted) wave. If the wave moves from a medium with a higher speed to one with a lower speed ($v_1 > v_2$) there is a 180° phase shift in the reflected wave. If it moves from lower to higher speed ($v_1 < v_2$) there is no phase shift. In the case of light a smaller speed corresponds to a larger index of refraction. The behavior of light under reflection can then be summarized.

$n_1 < n_2$	180° phase shift
$n_1 > n_2$	no phase shift

Thin Film Interference

Consider light normally incident on a thin film with index n of thickness t sitting on a medium with index n' . At the top interface there is always a phase shift of 180° . At the bottom surface there may or may not be a phase shift; if $n < n'$ there is a 180° phase shift and if $n > n'$ there is no phase shift. The interference is between the two reflected pulses, one of the top interface and one off the bottom.



The path difference is $2t$ and the wavelength in the film is λ/n . For the case of $n < n'$ the two phase shifts cancel. If the path difference is an integer number of wavelengths there is constructive interference and a half integer number of wavelengths corresponds to destructive interference. The other case of $n > n'$ adds a relative 180° phase shift between the two reflected rays and has the effect of swapping the conditions of constructive and destructive interference. The table summarizes this.

	Constructive Interference	Destructive Interference
$n < n'$	$2t = m \frac{\lambda}{n}$	$2t = \left(m + \frac{1}{2}\right) \frac{\lambda}{n}$
$n > n'$	$2t = \left(m + \frac{1}{2}\right) \frac{\lambda}{n}$	$2t = m \frac{\lambda}{n}$

Polarization

In discussing plane waves in Chapter J we found a solution of the form:

$$\vec{E} = \hat{y} E_{\max} \cos(kx - \omega t)$$

$$\vec{B} = \hat{z} B_{\max} \cos(kx - \omega t).$$

Here we have waves propagating in the x -direction with electric and magnetic fields in the y and z -directions, respectively. Generally for plane wave solutions the electric and magnetic fields are mutually perpendicular and both are perpendicular to the direction of propagation. There is a plane of possible directions perpendicular to the direction of propagation, representing different polarizations. We take the direction of the electric field to be the direction of polarization of the wave; this is the y -direction in the solution shown above.

Polarized Light through a Filter

A polarizing filter lets through only the component of the electric fields along the axis of polarization. Suppose polarized light is incident on a polarizing filter where the angle between the polarizing axis of the filter and the polarization of the light is θ . If $A_0 = E_{\max}$ is the amplitude of polarized light incident on a filter then

$$A = A_0 \cos \theta$$

is the amplitude of light leaving the filter. Since the intensity is proportional to the square of the amplitude $I \propto A^2$, we get

$$I = I_0 \cos^2 \theta$$

relating the intensities of the light before (I_0) and after (I) the filter. This relation is known as Malus's law.

Unpolarized Light through a Filter

Viewing normal ambient light through a polarizing filter usually shows no effect when the filter is rotated. This is because the light is unpolarized, meaning that it is a random mixture of all polarizations. Since the average value of \cos^2 is $1/2$ we get.

$$I = \frac{1}{2} I_0$$

relating the intensities before and after the filter.

Polarization by Reflection

Reflected light tends to be polarized. When light reflect of a surface at some angle (not normally incident) there is one possible polarization that is parallel to the surface. The reflected light tends to be polarized in this parallel direction. For instance, light reflecting off a horizontal surface tends to be horizontally polarized. Polarizing sunglasses have filters with vertical axes to remove this reflected light or glare.

It turns out that when the reflected light ray is perpendicular to the refracted ray, the reflected ray is *totally polarized*. A simple application of Snell's law gives that this occurs when the incident angle is $\theta_1 = \theta_p$ where θ_p , the polarizing angle or Brewster's angle, is

$$\tan \theta_p = \frac{n_2}{n_1}.$$