Chapter E

Current, Resistance and DC Circuits

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Current and Current Density

Basic Definitions

If dQ is the charge that passes through some surface, usually a cross-section of a wire, in time dt then the current I is defined by

$$I = \frac{dQ}{dt}$$

The current is related to the surface integral of a vector field, the current density $\vec{J}$, by

$$I = \int \vec{J} \cdot d\vec{A}$$

When the surface is the cross-section of a wire, the above expression becomes simply $I = J\,A$.

Drift Velocity

We can relate the current and current density to the flow of charge carriers. Let q be the charge of the charge carriers.

$$q = \text{charge of charge carriers}$$

The typical case is a metal, where the charge carriers are electrons and thus q is negative. Semiconductors can have positive or negative charge carriers. Define n as the density of charge carriers.

$$n = \frac{\text{\# of charge carriers}}{\text{Volume}}$$

The drift velocity is the bias in the motion of the charge carriers. Without a current the charge carriers are moving but there is no bias in the motion; their average velocity (vector) is zero. If there is an electric field in the conductor the charge carriers will move with a bias and the drift velocity is the average velocity of the charge carriers.

$$\vec{v}_d = \vec{v}_{\text{average}} = \text{drift velocity}$$

A useful analogy is the motion of gas molecules. The average velocity of gas molecules is zero, unless there is a wind, and then the average velocity is the wind velocity. The wind velocity is the analog of drift velocity.

Consider the simple case of current in a wire with cross-section A. In a time dt all the charge in the right cylinder with base A and height $v_d\,dt$ will pass the surface. This charge is

$$dQ = \frac{\text{charge}}{\text{Volume}} \times \text{Volume} = |q|\,n \times A\,v_d\,dt$$

The current is dQ/dt giving

$$I = |q|\,n\,A\,v_d.$$

This can be generalized to a vector expression for the current density

$$\vec{J} = q\, n\, \vec{v}_d.$$

For negative charge carriers the drift velocity is opposite to the current density.

Resistance

Ohm's Law

If there is an electric field in a conductor then there will be a current. We can define the conductivity and resistivity by the microscopic form of Ohm's law,

$$\vec{J} = \sigma\, \vec{E} \quad \text{(microscopic form)}$$

$$\sigma = \text{Conductivity}$$

$$\rho = \frac{1}{\sigma} = \text{Resistivity}$$

The conductivity and resistivity are properties of a material. For an object, like a wire, we can define a quantity called the resistance R by the macroscopic form of Ohm's law

$$V = I\, R \quad \text{(macroscopic form)}$$

$$R = \text{Resistance}$$

The resistance, which is a property of a wire, is related to the resistivity, which is a property of the wire's material. We want an expression for the resistance of a wire of length ℓ with cross-sectional area A. The electric field is related to the voltage and the length.

$$\Delta V = -\int \vec{E} \cdot d\vec{r} \implies V = |\Delta V| = E\, \ell$$

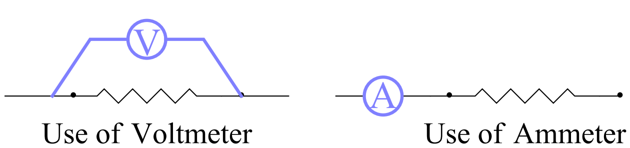
It follows that:

$$J = \sigma\, E \implies \frac{I}{A} = \frac{1}{\rho} \frac{V}{\ell}.$$

Ohm's law then gives

$$R = \frac{\rho\, \ell}{A}.$$

$V = I\, R$ relates the voltage *across* a resistor to the current *through* it. When passing through the resistor in the direction of the current, it is a voltage drop, a decrease in potential. To measure the voltage across a resistor connect the leads of the voltmeter to either side of the resistor. To measure the current through a resistor connect the ammeter in line with the resistor.



Variation of Resistance with Temperature

Resistance in a metal is caused by collisions between the moving electrons with the vibrating atoms. If there were no vibration in the atoms there would be no collisions and the resistance would be zero. As the temperature is increased the vibrational motion of the atoms increases and the collisions increase. This is why resistance increases with temperature. The increase of resistivity with temperature can be described by

$$\Delta\rho = \alpha\,\rho_0\,\Delta T \text{ or } \rho = \rho_0\,(1 + \alpha\,\Delta T).$$

Here α is defined as the temperature coefficient, which is a property of a material. ρ_0 is the resistivity at temperature T_0 and ρ is the resistivity at T. $\Delta T = T - T_0$ and $\Delta\rho = \rho - \rho_0$. Multiplying by ℓ / A gives expressions for the resistance

$$\Delta R = \alpha\,R_0\,\Delta T \text{ or } R = R_0\,(1 + \alpha\,\Delta T).$$

Power and DC Voltage Sources

Power in General

Power is generally defined as the time derivative of some energy or work

$$\mathcal{P} = \frac{d}{dt}\text{Energy}.$$

When a charge Q is moved across a potential difference ΔV the potential energy difference is $\Delta U = Q\,\Delta V$. It follows that when an infinitesimal charge dQ moves across a voltage of V the infinitesimal energy change is $dU = V\,dQ$. Writing $\mathcal{P} = dU/dt$ and using $I = dQ/dt$ gives

$$\mathcal{P} = V\,I.$$

Power Dissipated in a Resistor

Ohm's law $V = I\,R$ relates the voltage drop across a resistor to the current through it. Using it we can write equivalent expressions for the power dissipated in a resistor.

$$\mathcal{P} = V\,I = I^2\,R = \frac{V^2}{R}$$

The energy lost to resistance is dissipated as heat. This is called *Joule heating*.

Terminal Voltage

Treat every DC voltage source as an ideal voltage source with EMF (electromotive Force) $\mathcal{E}$ in series with its internal resistance r. The voltage across the terminals V_t of the source is then

$$V_t = \mathcal{E} - I\,r.$$

When there is no load, $I = 0$, the terminal voltage V_t is the same as the EMF $\mathcal{E}$. With a load the terminal voltage drops.

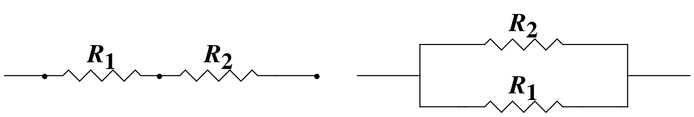
Circuit Diagrams and Nodes

A real-world wire has resistance. When we draw circuit diagrams we always consider the wires to be perfect conductors. Since $R = 0$, the voltage drop across a wire is zero. A wire in a circuit diagram is a point of constant voltage; this is what we call a node. The most effective way to analyze complex circuit diagams is in terms of nodes and the circuit elements (voltage sources, resistors, capacitors, etc.) connected between nodes.

If it is necessary to consider the real-world resistance of a wire can can simply view it as an ideal conductor with a resistor with $\rho\,\ell/A$ of resistance placed in line.

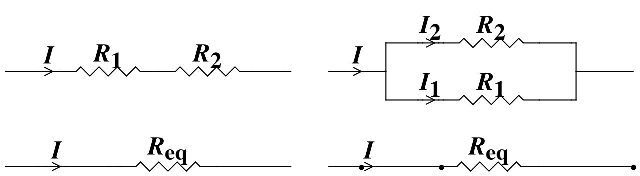
Voltages in circuits are always differences. If we choose some node to be zero voltage then we can assign a voltage to each node in a circuit. A point of zero voltage in a circuit is called a *ground*.

Combinations of Resistors



Series Parallel

Any combination of resistors with one wire in and one wire out can be reduced to its equivalent resistance. If the combination were placed inside some black box then outside the box the combination would look like a single resistor, which we call its equivalent resistance. For series and parallel resistor combinations, there are simple formulas for finding these equivalent resistances.



Series Parallel

Series

Resistors are in series when all the current through one passes through the others; there is no branching between them. The total voltage is the sum of the voltages.

$$I = I_1 = I_2 = \ldots \quad \text{and} \quad V = V_1 + V_2 + \ldots$$

Using $V = I\,R$ gives $I\,R_{\text{eq}} = I\,R_1 + I\,R_2 + \ldots$. The equivalent resistance of series resistors is given by

$$R_{\text{eq}} = R_1 + R_2 + \ldots.$$

Parallel

Resistors are in parallel when the voltage across the one is the same as the voltage across the others. Resistors are in parallel when they are connected between the same two nodes, where a node is a point of constant voltage in a circuit. The current branches and the total current is the sum of the currents.

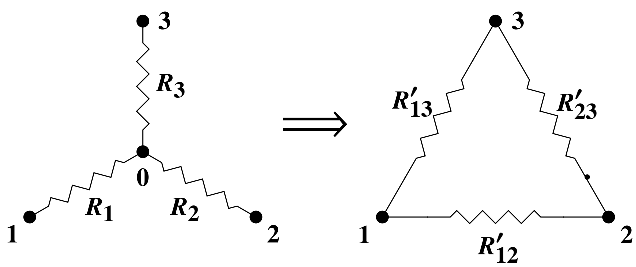
$$V = V_1 = V_2 = \ldots \quad \text{and} \quad I = I_1 + I_2 + \ldots$$

Using $I = V/R$ gives $V\big/R_{\text{eq}} = V/R_1 + V/R_2 + \ldots$. The equivalent resistance of series resistors is given by

$$R_{eq} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \ldots\right)^{-1}.$$

Node Reduction

Not all resistor networks can be reduced to their equivalent resistance using the series and parallel rules mentioned above. For these networks another approach is needed. The diagram below shows the idea of this method. Where three resistors R_1, R_2 and R_3 diverge from a central node, marked node 0 in the diagram, we can replace these three with three other resistors R'_{12}, R'_{13} and R'_{23}.



The new resistances can be related to the old ones by the simple expression

$$R'_{ij} = \frac{R_i R_j}{R_{||}} \text{ where } R_{||} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-1}$$

Node Reduction - General *N* Resistor Case

The formula above generalizes to N resistors leaving a point. Call the resistors R_i with $i = 1, \ldots, N$. They connect a common node labeled by 0 with an external node labeled by i. Now we replace the N resistors with ones connecting all possible pairs of the N external nodes. The new resistances have the values

$$R'_{ij} = \frac{R_i R_j}{R_{||}} \text{ where } R_{||} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \ldots\right)^{-1}.$$

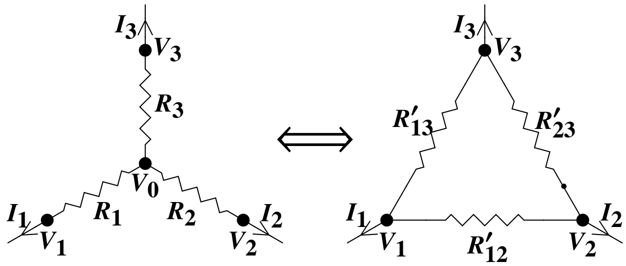
The $N = 2$ case is just two resistors in series and the series formula is a special case of this.

$$N = 2 \implies R_{||} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1} = \frac{R_1 R_2}{R_1 + R_2} \implies R'_{12} = \frac{R_1 R_2}{R_{||}} = R_1 + R_2$$

Summarizing the number of new resistances we have this table.

Number of R_i	Number of R'_{ij}
2	1
3	3
4	6
N	$\frac{1}{2} N (N-1)$

Proof of Node Reduction Formula



For clarity, we will consider the proof of the formula for the three resistor case but the following derivation can easily be modified to prove the formula in the general case. Take the voltages at each node to be V_1, V_2, V_3 and V_0. Take the outward (away from 0) current through R_i as I_i. It follows that

$$I_1 = \frac{V_0 - V_1}{R_1},\ \ I_2 = \frac{V_0 - V_2}{R_2} \text{ and } I_3 = \frac{V_0 - V_3}{R_3}.$$

The condition that the currents sum to zero gives V_0.

$$I_1 + I_2 + I_3 = 0 \implies V_0\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) = \frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3}$$

$$\implies V_0 = R_{||}\left(\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3}\right) \text{ where } R_{||} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-1}$$

Using this value for V_0 we can find the current as a function of voltage

$$I_1 = \frac{V_0}{R_1} - \frac{V_1}{R_1} = \frac{R_{||}}{R_1}\left(\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3}\right) - \frac{V_1}{R_1}.$$

Multiply the term on the right by one, in the form $R_{||}/R_{||}$, then cancel terms and regroup.

$$I_1 = \frac{R_{||}}{R_1}\left(\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3}\right) - \frac{V_1}{R_1} \times R_{||}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$= \frac{R_{||}}{R_1}\left(\frac{V_2 - V_1}{R_2} + \frac{V_3 - V_1}{R_3}\right) = \frac{V_2 - V_1}{R'_{12}} + \frac{V_3 - V_1}{R'_{13}}$$

where we have used

$$R'_{ij} = \frac{R_i R_j}{R_{||}}.$$

This gives the simple result

$$I_1 = \frac{V_2 - V_1}{R'_{12}} + \frac{V_3 - V_1}{R'_{13}}.$$

Clearly, there is nothing special about I_1 in the above derivation and we can derive similar results for I_2 and I_3.

$$I_2 = \frac{V_1 - V_2}{R'_{12}} + \frac{V_3 - V_2}{R'_{23}} \text{ and } I_3 = \frac{V_1 - V_3}{R'_{13}} + \frac{V_2 - V_3}{R'_{23}}$$

The above relations prove our result. It shows the current to voltage relation is the same for the 3 resistances R_1, R_2 and R_3 leaving the 0 node as for the replacement resistances R'_{12}, R'_{13} and R'_{23} connecting the external nodes.

The Equivalent Resistance Theorem

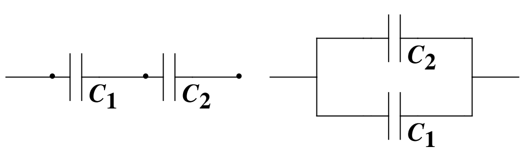
Any network of resistors with two external nodes may be reduced to a single equivalent resistance between the external nodes. An algorithm for finding the equivalent resistance of a network follows.

Specify the resistor network with a set of nodes, two external and the rest internal, and with a set of resistors, where each resistor is labeled by a resistance *and* by the pair of nodes it connects.

The first step is to remove all resistances in parallel; these are resistors between the same two nodes. The second step is to apply the node reduction procedure to any of the internal nodes. Continue iterating these two steps until all internal nodes are removed and there is a single resistor.

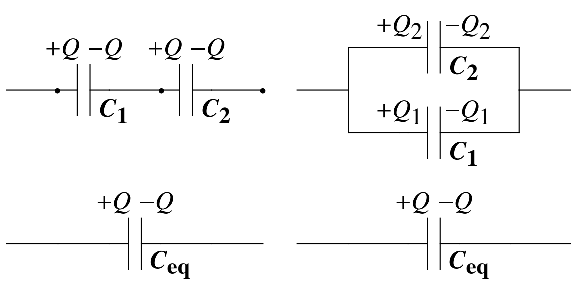
To make this most calculationally efficient, remove an internal node with the smallest number of resistances. First look for a pair in series (or $N = 2$.) If none are in series then look for $N = 3$, then $N = 4$, etc.

Combinations of Capacitors



Series Parallel

As we saw for resistors, any network of capacitors can be reduced to an equivalent capacitance. For capacitors its charge plays the role the current played in resistors. (Recall that $I = dQ/dt$.)



Series Parallel

The voltage to charge relation for a capacitor is

$$V = \frac{Q}{C}.$$

Series

In the case of series resistors the charge on each capacitor is the same and both are the same as the charge on the equivalent. The voltages add.

$$Q = Q_1 = Q_2 = \ldots \quad \text{and} \quad V = V_1 + V_2 + \ldots$$

Using the voltage to charge relation gives $Q/C_{\text{eq}} = Q/C_1 + Q/C_2 + \ldots$ which gives the expression for equivalent capacitance

$$C_{\text{eq}} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \ldots\right)^{-1}.$$

Parallel

For parallel resistors the voltages are equal and the charges add.

$$V = V_1 = V_2 = \ldots \quad \text{and} \quad Q = Q_1 + Q_2 + \ldots$$

Using $Q = C\,V$ gives $C_{\text{eq}}\,V = C_1\,V + C_2\,V + \ldots$ giving

$$C_{\text{eq}} = C_1 + C_2 + \ldots .$$

Note that the series and parallel formulas for capacitors are reversed relative to their resistor counterparts.

Node Reduction

The node reduction formula also applies to capacitors as well. The new capacitors have the values

$$C'_{ij} = \frac{C_i\,C_j}{C_{||}}$$

but the $C_{||}$ has a different form.

$$C_{||} = C_1 + C_2 + \ldots$$

Kirchhoff's Rules

Kirchhoff's rules are used to solve for the currents in the case of a circuit involving many resistors and DC voltage sources. A junction is a point in the circuit where three or more wires meet; if there are just two wires it is just a bend in the wire and not a junction. For every branch in the circuit we can define a current. Kirchhoff's rules gives a set of linear equations in the currents. It is not essential to choose the proper direction for the currents, and in fact one typically doesn't know the current directions until a solution is found. If the chosen current direction is wrong then that current will be negative when the solution is found.

Junction Rule

At every junction in a circuit the total current in is equal to the total current out.

$$\sum I_{\text{in}} = \sum I_{\text{out}}.$$

In every case (at least where the circuit is one connected piece) the junction rule equations will not be independent; there will always be one equation more than is needed. Summing oll the equations gives $\sum I = \sum I$ which is equivalent to $0 = 0$. (This is because every current leaves one junction and enters another.) Because of this any one junction rule equation is the negative of the sum of the others. To get an independent set of equations one must delete one (any one) of the equations.

Loop Rule

Around every closed loop in a circuit the sum of all the voltage gains is zero.

$$\sum \Delta V = 0$$

The sign conventions are:

When moving through a resistor in the direction of the current: $\Delta V = -I\,R$.
When moving through a resistor opposite the current: $\Delta V = +I\,R$.

When moving through a DC source from - to + terminals: $\Delta V = +\mathcal{E}$.
When moving through a DC source from + to - terminals: $\Delta V = -\mathcal{E}$.

To avoid non-independent equations consider only the smallest loops.