Chapter D

Capacitance

Blinn College - Physics 2426 - Terry Honan

A capacitor is just any configuration involving two conductors. Begin with the two conductors neutral and then connect some DC voltage source (a battery, for instance) with a voltage $V = |\Delta V|$ across them. Charge will flow until an electrostatic state is reached. The conductor connected to the positive terminal will gain some charge $+Q$ and the other will gain charge $-Q$. There is a proportionality between charge and voltage. To see this write the potential in terms of some charge distribution $V = k_e \int dq/r$; if we double all the charge $dq \to 2\,dq$ then the voltage will double. Define the capacitance as the constant of proportionality between a charge and voltage

$$Q = C\,V \quad (C \text{ is the capacitance.})$$

The capacitance between any pair of conductors depends on the geometry of the conductors, meaning that it depends on their size, shape, relative orientation and relative distance. We can measure the capacitance for any configuration but we can only calculate it in simple cases of symmetry.

Calculations of Capacitance

The symmetric cases we will consider will involve the tree symmetries we used in Gauss's law: spherical, cylindrical and planar symmetry. The general procedure we will use is:

$$Q \xrightarrow{\text{Gauss's law}} \vec{E} \xrightarrow{\Delta V = -\int \vec{E}\cdot d\vec{r}} V$$

We will assume here that there is a vacuum between the conductors. Later we will see that adding a medium between the conductors will enhance the capacitance. We will denote the empty capacitance by C_0.

The Parallel Plate Capacitor

There is a pair of parallel plates with cross-sectional area A separated by d. We assume that d is small compared to the smallest linear dimension in A. For instance, if the cross-section is circular the separation is small compared to the radius and if rectangular it is small compared to the smaller of the length and width. The electric field is perpendicular to the plate and the equipotentials are parallel to the plates.

Using Gauss's law we related the field at the surface of a conductor to the surface charge density

$$E = \frac{\sigma}{\varepsilon_0} \text{ and } \sigma = \frac{Q}{A} \implies E = \frac{Q}{\varepsilon_0 A}.$$

The voltage is the potential difference. Consider a path $\Delta \vec{r}$ between the plates in the direction opposite to the field.

$$V = \Delta V = -\vec{E}\cdot\Delta\vec{r} = E\,d = \frac{Q}{\varepsilon_0 A}\,d$$

Using the definition of capacitance $Q = C_0\,V$ we can find C_0.

$$C_0 = \frac{\varepsilon_0 A}{d}$$

The Spherical Capacitor

A spherical capacitor consists of an inside conducting sphere of radius a sitting inside an outside conducting sphere with an inside radius b, concentric with the first surface. (Note that the outside radius of the conductor doesn't matter.) Connect the inside conductor to the positive terminal of the DC voltage source and the negative terminal to the outside conductor. The inside conductor gains charge $+Q$ and the outside gains $-Q$.

First we use Gauss's law to give the field from the charge. For any case of spherical symmetry Gauss's law gives $\vec{E} = k_e Q_{\text{inside}} \frac{\hat{r}}{r^2}$. For $a < r < b$ we get $Q_{\text{inside}} = Q$. The voltage of the source, V, will be the potential difference when moving from b to a. Choose the contour of integration of the potential to be along a radius from b to a.

$$V = \Delta V = -\int \vec{E} \cdot d\vec{r} = -\int_b^a k_e \frac{Q}{r^2}\, dr = k_e Q \left(\frac{1}{a} - \frac{1}{b} \right)$$

The definition of capacitance $Q = C_0 V$ gives C_0.

$$C_0 = \frac{1}{k_e \left(\frac{1}{a} - \frac{1}{b} \right)}$$

The Cylindrical Capacitor

A cylindrical capacitor consists of a long cable of length ℓ with an inside cylindrical conductor of radius a and an outside cylinder with an inside radius of b. Once again we will connect the inside conductor to the positive terminal and the outside to the negative. The inside gets charge $+Q$ and the outside gains $-Q$.

Gauss's law for cylindrical symmetry gives $\vec{E} = \frac{1}{2\pi\varepsilon_0} \frac{Q_{\text{inside}}/\ell}{r} \hat{r}$. As with the spherical case, for $a < r < b$ we get $Q_{\text{inside}} = Q$. and we will integrate radially inward from b to a.

$$V = \Delta V = -\int \vec{E} \cdot d\vec{r} = -\int_b^a \frac{1}{2\pi\varepsilon_0} \frac{Q/\ell}{r}\, dr = \frac{Q/\ell}{2\pi\varepsilon_0} \ln \frac{b}{a}$$

Solving for the capacitance gives

$$C_0 = \frac{2\pi\varepsilon_0 \ell}{\ln(b/a)}.$$

Capacitance of a Single Sphere

We have defined capacitance as a property of two conductor but we can define it for a single conductor as well. To do this imagine the second conducting surface to be a sphere at infinity. As an example, consider a single conducting sphere of radius R. Inserting $a = R$ and $b = \infty$ into the spherical capacitor formula gives the capacitance of a single sphere.

$$C_0 = \frac{R}{k_e}$$

Energy

Energy in Capacitor

Vary the charge on the plates from 0 to Q, $0 \le q \le Q$. The voltage as a function of charge is $V(q) = q/C$. When a charge dq is moved across a voltage V the change in energy is $dU = V(q)\,dq$. Integrating gives

$$U = \int_0^Q V(q)\,dq = \int_0^Q \frac{q}{C}\,dq = \frac{Q^2}{2C}.$$

Since $Q = C\,V$ we can rewrite the expression as

$$U = \frac{1}{2} C\,V^2 = \frac{Q^2}{2C} = \frac{1}{2} Q\,V.$$

Energy in an Electric Field

The energy in a capacitor is stored in the electric field between the plates. In general, wherever there is an electric field there is energy stored in that field. The energy density u is the energy per volume. We will derive an expression for the energy density in a field by using what we know about capacitors. In a parallel plate capacitor $C_0 = \frac{\varepsilon_0 A}{d}$ and $V = E\,d$ giving the energy is

$$U = \frac{1}{2} C_0\,V^2 = \frac{1}{2} \frac{\varepsilon_0 A}{d} (E\,d)^2.$$

The volume between the plates is $A\,d$ giving $u = \frac{U}{A\,d}$ and thus

$$u = \frac{1}{2} \varepsilon_0\,E^2.$$

Self-Energy of a Point Charge

We can calculate the energy in the field of a point charge. It is a surprising and somewhat disturbing fact that this self-energy of a point charge is infinite. To calculate this we will consider a conducting sphere of radius δ. This will be finite and the point charge will be the $\delta \to 0$ limit. The magnitude of the field for a conducting sphere with charge Q is

$$E = k_e \frac{Q}{r^2} \quad \text{for } r > \delta$$

$$E = 0 \quad \text{for } r < \delta$$

The total energy in the field can be found by integrating the energy density over all of space. Since the field varies only with r we can integrate over concentric spheres. The volume element is $d\text{Volume} = 4\,\pi\,r^2\,dr$.

$$
\begin{aligned}
U_\delta(Q) &= \int u \, d\text{Volume} = \frac{\varepsilon_0}{2} \int E^2 \, d\text{Volume} \\
&= \frac{\varepsilon_0}{2} \int_\delta^\infty \left(k_e \frac{Q}{r^2} \right)^2 4 \pi r^2 \, dr = \frac{k_e Q^2}{2} \int_\delta^\infty \frac{dr}{r^2} \\
&= \frac{k_e Q^2}{2 \delta}
\end{aligned}
$$

An alternative and simpler derivation of this is to consider the sphere a capacitor with charge Q.

$$C_0 = \frac{\delta}{k_e} \text{ and } U = \frac{Q^2}{2 C_0} \Longrightarrow U_\delta(Q) = \frac{k_e Q^2}{2 \delta}$$

It is clear from the above expression that the energy in the field of a point charge is infinite

$$\lim_{\delta \to 0} U_\delta(Q) = \infty.$$

How can this make sense? Recall that with potential energy in general, the zero of energy is arbitrary. Energy differences are the important quantities. Let us consider the energy difference between two point charges Q_1 and Q_2 a distance r apart and the self-energies of the two charges separately. To keep things finite, consider two small conducting spheres of radius δ separated by r. Defining $U_\delta(Q_1 , Q_2 ; r)$ as the energy of two conducting spheres of radius δ with charges Q_1 and Q_2 separated by r as a distance between their centers. It is beyond the scope of this class to calculate this energy, $U_\delta(Q_1 , Q_2 ; r)$, but proper calculation shows that it is also infinite in the $\delta \to 0$ limit. It can also be shown that the self energies of the two charges is subtracted from the energy of the charges at a distance r we get: a result which is finite in the $\delta \to 0$ limit

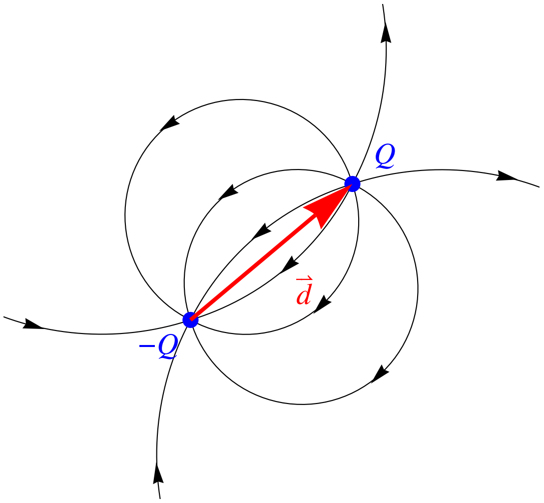
$$\lim_{\delta \to 0} \left[U_\delta(Q_1 , Q_2 ; r) - U_\delta(Q_1) - U_\delta(Q_2) \right] = k_e \frac{Q_1 Q_2}{r.}$$

This is just the potential energy of a two charge configuration.

Electric Dipoles

An electric dipole is some charge configuration with a net separation of charge but zero net charge. We saw the field for a dipole in Chapter A. Here we want to quantify the strength of a dipole by defining an electric dipole moment $\vec{p}$.

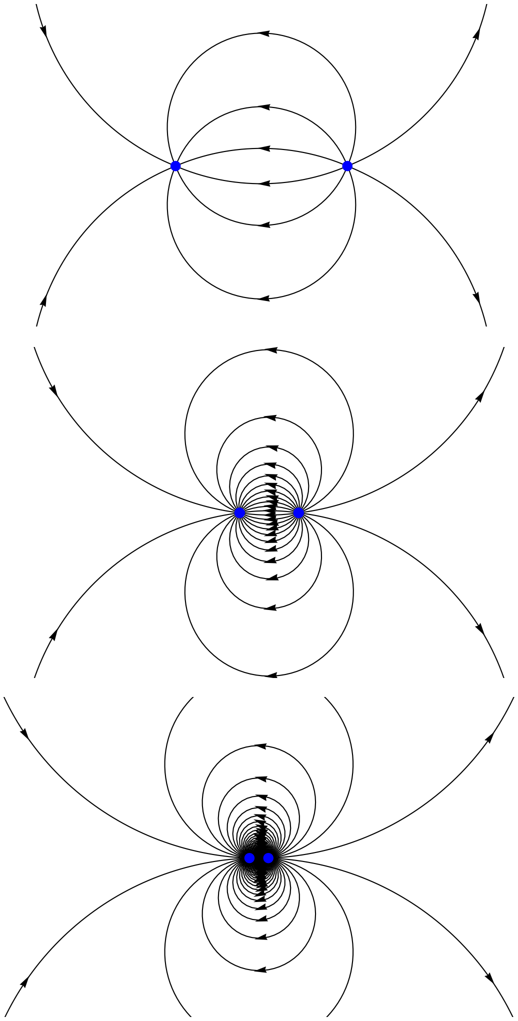
Electric Dipole Moment



There are two charges $+Q$ and $-Q$. Define the vector pointing from the negative charge to positive charge as $\vec{d}$. Define the electric dipole moment by

$$\vec{p} = Q\,\vec{d}.$$

Note that if the charge is doubled while the distance is halved the dipole moment stays fixed. If we take the limit as the distance d goes to zero while keeping p fixed, so Q becomes infinite, we get a point dipole. Many molecules have dipole moments. The field diagrams show different configurations with the same dipole moments but with decreasing d values.



For instance, in a water molecule the electrons from the hydrogen tend to spend time around the oxygen; this makes the oxygen side negative and the hydrogen side is positive. A water molecule has some measurable dipole moment.

When placed in an electric field there is a force on the positive charge in the direction of the field and a force on the negative charge opposite the field. This creates a torque causing the dipole to rotate until aligned with the field.

Torque and Potential Energy

An electric dipole in some uniform electric field will give net force, but there is a net torque. The torque at some position $\vec{r}$ is given by $\vec{\tau} = \vec{r} \times \vec{F}$. Torque depends on the choice of origin, however the net torque in the case of zero net force is independent of the choice of origin. Here we want the net torque on the dipole.

$$\begin{aligned}\vec{\tau} &= \sum \vec{\tau} = \vec{r}_+ \times \vec{F}_+ + \vec{r}_- \times \vec{F}_- = \vec{r}_+ \times \left(Q\vec{E}\right) + \vec{r}_- \times \left(-Q\vec{E}\right) \\ &= \left(\vec{r}_+ - \vec{r}_-\right) \times Q\vec{E} = \vec{d} \times Q\vec{E}\end{aligned}$$

The definition of the dipole moment gives

$$\vec{\tau} = \vec{p} \times \vec{E}.$$

The torque becomes zero when the dipole is aligned with the field. This is the equilibrium position.

The equilibrium position is the position of lowest potential energy. To get an expression for the potential energy fix the position of the negative charge and rotate the positive charge on a circle of radius d. Since the force on the positive charge is constant the change in potential energy is given by

$$\Delta U = -\vec{F} \cdot \Delta \vec{r} = -F\, \Delta r_{\parallel},$$

where $\Delta r_{\parallel}$ is the component of $\Delta \vec{r}$ in the direction of the field. If we define the zero of potential energy to be where the $\vec{d}$ vector is perpendicular to the field then the $\Delta r_{\parallel}$ measured from the zero position is given by $\Delta r_{\parallel} = d \cos\theta$, where θ is the angle between the dipole and the field.

$$U = -F\, d \cos\theta = -\vec{F} \cdot \vec{d} = -Q\, \vec{E} \cdot \vec{d}$$

Using the definition of the dipole moment gives

$$U = -\vec{E} \cdot \vec{p}.$$

Dielectrics

A dielectric is a medium placed between the conductors of a capacitor. A dielectric will enhance capacitance. It is the polar nature of the molecules in the dielectric that create this effect and a good dielectric is one with strongly polar molecules.

Begin with a parallel plate capacitor with a uniform field magnitude E_0 between the plates. The field is related to the charge and charge density between the plates by

$$E_0 = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0\, A}.$$

If a conducting slab is placed between the plates the field inside the conductor will be $E = 0$. For this to happen charge in the conductor moves to the edges to cancel the charge in the capacitor. If a dielectric is placed between the plates the alignment of the dipoles is equivalent to a partial shielding of the capacitors charge and the field in the slab is diminished to a smaller value $E < E_0$. The dielectric constant κ is defined as the constant $\kappa \geq 1$ given by

$$E = \frac{E_0}{\kappa}.$$

If the slab is added fill the entire region between the plates while the charge on the plates is held constant. (This is done by disconnecting the capacitor from any voltage source.)

$$Q = Q_0$$

Voltage is related to the field by the standard relation $\Delta V = -\int \vec{E} \cdot d\vec{r}$; this implies that the voltage changes as the slab is added.

$$V = E\, d \text{ and } V_0 = E_0\, d \implies V = \frac{V_0}{\kappa}$$

$$C = \frac{Q}{V} \text{ and } C_0 = \frac{Q_0}{V_0} \implies C = \kappa\, C_0$$

This is the main result; adding a dielectric between the conductors enhances the capacitance by a factor called the dielectric constant κ. The dielectric constant is a material dependent constant