Chapter C

Potential and Potential Energy

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First some terminology: We will define electric potential V as the potential energy per test charge, $V = U/q_0$. The similarity of the letters U and V combined with the similar names, potential versus potential energy, creates a confusion between these related but distinct notions.

Voltage is potential difference ΔV; that is, differences in potential. When the sign of ΔV is ignored, we will sometimes use the standard, somewhat ambiguous, notation where voltage is written as just V, instead of the more precise $|\Delta V|$.

Basic Definitions

$\vec{F}$	$\longleftarrow$ $\vec{E} = \vec{F}/q_0$ $\longrightarrow$ $\vec{F} = Q\,\vec{E}$	$\vec{E}$
$\uparrow$		$\uparrow$
$\Delta U = -\int \vec{F} \cdot d\vec{r}$		$\Delta V = -\int \vec{E} \cdot d\vec{r}$
$\downarrow$		$\downarrow$
U	$\longleftarrow$ $V = U/q_0$ $\longrightarrow$ $\Delta U = Q\,\Delta V$	V

In general one defines work done by a force as the integral of the force along a contour, $W = \int \vec{F} \cdot d\vec{r}$. When the work done by a force is independent of path, meaning that it depends only on the endpoints of the path, we say the force is conservative. For any conservative force we can define a potential energy U by $\Delta U = -W$ giving:

$$\Delta U = -\int \vec{F} \cdot d\vec{r}.$$

Recall that we define the electric field by considering the force on a test charge and by dividing the charge into the force, $\vec{E} = \vec{F}/q_0$. The potential is defined from the potential energy similarly. The potential is defined as the potential energy per charge

$$V = \frac{U}{q_0}.$$

The zero of potential energy is arbitrary and so is the zero of voltage. When a charge Q is moved across a potential difference ΔV we get a change in potential energy given by

$$\Delta U = Q\,\Delta V.$$

If the above expressions are taken as the definition of V then dividing both sides of the ΔU formula gives an expression for the voltage (potential difference) as an integral of the electric field over a contour

$$\Delta V = -\int \vec{E} \cdot d\vec{r}.$$

This is the fundamental expression showing how potential can be found from electric fields.

Potential for Charge Distributions

We have found expressions for electric fields due to charge distributions, first considering the point charge and then discrete and continuous distributions. We will now do the same for potential. Since potential is a scalar we will see that the potential calculations are simpler because they lack the vector complications in the field calculations.

Point Charge

We know that the field of a point charge is $\vec{E} = k_e\, Q\, \frac{\hat{r}}{r^2}$ and the general expression for the potential difference from a field. Combining the two gives and integral for the field for a point charge.

$$\Delta V = -\int \vec{E} \cdot d\vec{r} = -k_e\, Q \int_{\vec{r}_i}^{\vec{r}_f} \frac{\hat{r}}{r^2} \cdot d\vec{r}$$

Multiplying a vector by a unit vector gives the vector component in the unit vector's direction. $\hat{r} \cdot d\vec{r}$ is just the radial component of $d\vec{r}$ which is just dr, the change in the radial variable

$$\hat{r} \cdot d\vec{r} = dr.$$

The limits of the integral then only depend only on the radial distances of the endpoints.

$$\Delta V = -k_e\, Q \int_{r_i}^{r_f} \frac{dr}{r^2} = k_e\, Q \left(\frac{1}{r_f} - \frac{1}{r_i} \right),$$

We are looking for some function $V(r)$, describing the potential as a function of position. Since it must satisfy $\Delta V = V(r_f) - V(r_i)$, it is unique up to an arbitrary constant. The simplest choice is

$$V(r) = k_e\, \frac{Q}{r}$$

where the arbitrary constant is chosen to make the potential zero at infinity

$$V(\infty) = 0 \quad \text{or more precisely} \quad \lim_{r \to \infty} V(r) = 0.$$

Discrete Distribution

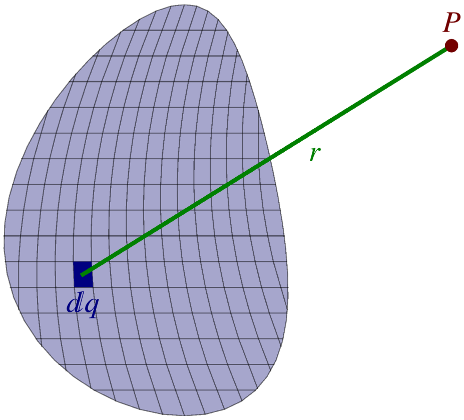
To find the voltage at a point P due to charges Q_i we will take r_i to be the distance from Q_i to P. The potential is

$$V = k_e \sum_i \frac{Q_i}{r_i}$$

Continuous Distribution

As was done in the case of the electric field we will take r to be the distance from dq to P. The potential is

$$V = k_e \int \frac{dq}{r}.$$



Interactive Figure

Potential Energy of Charge Distributions

We have seen that the potential energy difference when a charge Q is moved across a potential difference ΔV is

$$\Delta U = Q\,\Delta V.$$

To find the potential energy of a configuration we will start with zero potential energy when all the charges are far apart. For a two charge configuration begin with Q_1 in place and move charge Q_2 from infinity to its position a distance r from Q_1. The potential due to Q_1 is varying from 0 to $k_e \frac{Q_1}{r}$.

$$U = \Delta U = Q_2 \left(k_e \frac{Q_1}{r} - 0 \right)$$

This gives the expression for two charges separated by a distance r

$$U = k_e \frac{Q_1 Q_2}{r}.$$

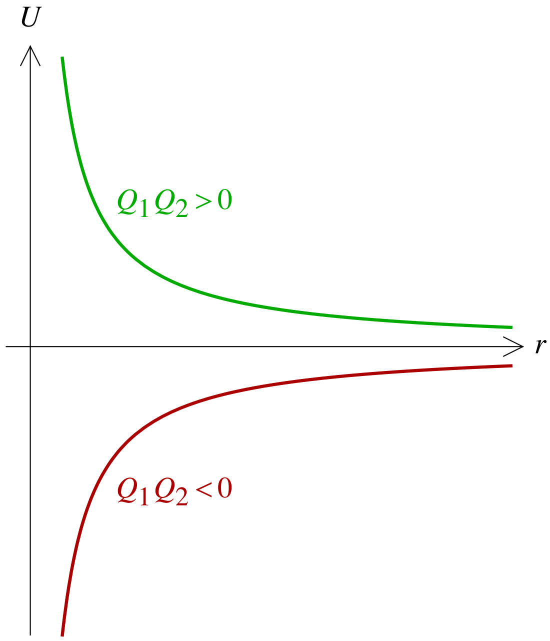


Figure - Potential Energy for Two Point Charges

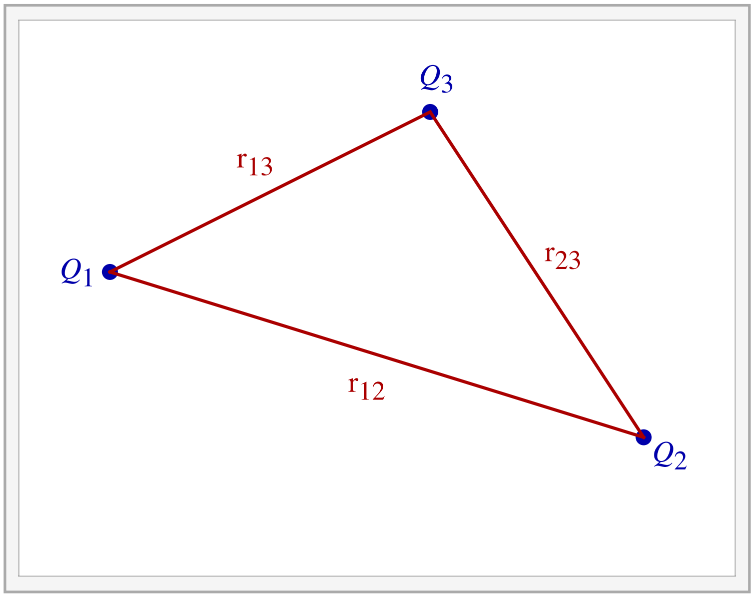
For a three charge configuration we have potential energy between each pair. Taking the distance between charges i and j to be r_{ij} gives the potential energy as

$$U = k_e \frac{Q_1 Q_2}{r_{12}} + k_e \frac{Q_1 Q_3}{r_{13}} + k_e \frac{Q_2 Q_3}{r_{23}}.$$

The general expression is

$$U = k_e \sum_{i<j} \frac{Q_i Q_j}{r_{ij}},$$

where the sum is over all pairs of charges. Note that keeping $i < j$ avoids double counting and excludes $i = j$, which would correspond to the energy between a particle and itself.



Interactive Figure

Potentials and Electric Fields

Potential from Electric Field

We have seen that the fundamental expression that gives the potential difference from the electric field is

$$\Delta V = -\int \vec{E} \cdot d\vec{r}.$$

This is the change in the potential between the endpoints of the contour of the integral. A simple special case of this is a uniform field. If $\vec{E}$ is uniform (meaning spatially constant) then it comes out of the integral.

$$\Delta V = -\int \vec{E} \cdot d\vec{r} = -\vec{E} \cdot \int d\vec{r} = -\vec{E} \cdot \Delta \vec{r},$$

where $\Delta \vec{r}$ is the vector from the starting position to the final position.

Another important special case is the infinitesimal form; if one moves along a small displacement $d\vec{r}$, the infinitesimal change in the potential is given by

$$dV = -\vec{E} \cdot d\vec{r}.$$

Equipotentials

Equipotentials are surfaces of constant potential.

Potential is a scalar field, meaning that at each position in space there is a scalar function. A way of representing scalar fields in a two dimensional graph is to draw contour of constant values. On weather maps, the scalars of temperature and pressure are represented by isotherms and isobars, which are contours of constant temperature and pressure. In three dimensions we have surfaces of constant values instead of contours. A surface of constant potential is called an equipotential and these will appear as contours in two dimensional diagrams.

Field lines are perpendicular to equipotentials.

To understand how equipotentials are related to electric field lines consider the infinitesimal form of the potential expression $dV = -\vec{E} \cdot d\vec{r}$. Take $d\vec{r}$ to be some infinitesimal displacement along the equipotential. Along the equipotential $dV = 0$, this is because the potential is constant along an equipotential. We then get $0 = \vec{E} \cdot d\vec{r}$ which implies that the field is perpendicular to the $d\vec{r}$. Since this is true for any direction along the equipotential it follows that the field is always perpendicular to the equipotential.

Electric field lines point toward lower potential.

Begin with the expression $dV = -\vec{E} \cdot d\vec{r}$ and take $d\vec{r}$ to be some infinitesimal displacement in the direction of an electric field line. Since the dot product of two vectors in the same direction is positive $\vec{E} \cdot d\vec{r} > 0$ which implies that $dV = -\vec{E} \cdot d\vec{r} < 0$. This tells us that field lines always point toward lower electric potential.

Electric Field from Potential

To go from the potential to the electric field, begin with the infinitesimal expression $dV = -\vec{E} \cdot d\vec{r}$. To get E_x consider some infinitesimal displacement in the x-direction $d\vec{r} = \hat{x}\, dx$.

$$dV = -\vec{E} \cdot d\vec{r} \quad \text{and} \quad d\vec{r} = \hat{x}\, dx \implies dV = -E_x\, dx$$

Solving for E_x gives $E_x = -\frac{dV}{dx}$. This expression is not strictly correct, because the potential may not only be a function of just x. By moving in the x-direction we are really taking the derivative with respect to x keeping the other variables (y and z) constant. This is what is meant by partial derivatives and the expressions for the components of the field are

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}.$$

The radial part of the field can also be found in a similar manner $E_r = -\frac{\partial V}{\partial r}$.

It should be mentioned as an *aside* that the above relations can be written as a single vector expression using the gradient operator of vector calculus.

$$\vec{E} = -\vec{\nabla} V$$

Conductors in Electrostatics - II

Potential is constant throughout a conductor in electrostatics.

The condition that the electric field is zero implies that the potential is constant. Consider some contour entirely inside a conductor. Integrating the field to get the potential difference gives

$$\Delta V = -\int \vec{E} \cdot d\vec{r} = 0.$$

Note that this argument does not apply to disconnected conductors. If there are several disconnected conductors each one will have its own potential.

It follows that the surface of a conductor is always an equipotential. The condition that the field is perpendicular to a conductor may now be seen as a consequence of fields being perpendicular to equipotentials.