Chapter B

Gauss's Law

Blinn College - Physics 2426 - Terry Honan

Electric Flux

Electric flux is a measure of the number of field lines passing through a surface. We will develop a definition of flux gradually, starting with special cases and generalizing.

Flux of a Uniform Field through a Flat Surface

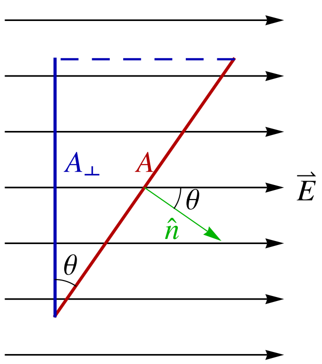
We saw in the discussion of electric field diagrams that the number of field lines per area is a measure of the strength of the field. Strictly, by area we mean $A_\perp$, the part of the area perpendicular to the field.

$$E \propto \frac{\#\text{ of lines}}{A_\perp}.$$

We want the flux Φ to be a measure of the number of lines through a surface $\Phi \propto (\#\text{ of lines})$ so we can define the flux in the case of a uniform field and a flat surface to be

$$\Phi = E\,A_\perp = E\,A\cos\theta.$$

The angle θ is measured between the electric field and the normal (perpendicular) to the surface.



Recall the dot product. For any two vectors $\vec{A}$ and $\vec{B}$, their dot product is given by

$$\vec{A}\cdot\vec{B} = A\,B\cos\theta = A_x\,B_x + A_y\,B_y + A_z\,B_z.$$

The expression for flux can then be written in terms of the dot product

$$\Phi = \vec{E}\cdot\vec{A},$$

where the vector $\vec{A}$ is defined to have magnitude A, the area and to be in the direction $\hat{n}$, the unit vector normal to the surface.

$$\vec{A} = A\,\hat{n}$$

Note that there are two unit normals to any flat surface; if $\hat{n}$ is a unit normal, then so is $-\hat{n}$. This gives a sign ambiguity in the flux. This, we will see, is a general feature, except in the case of a closed surface where we can choose the outward normal.

Flux in General

The expression for flux must now be generalized to the case of a general field that varies spatially and a surface that is not flat. To do this, break the surface into many small (infinitesimal) flat pieces with area vectors given by $d\vec{A} = \hat{n}\, dA$, where dA is the infinitesimal area of the surface and $\hat{n}$ is the unit normal at that position. For sufficiently small pieces the field will be uniform to a good approximation. For one such flat piece, the infinitesimal flux is $\vec{E} \cdot d\vec{A}$. Summing over all the small flat pieces gives an integral. The general definition of flux is then:

$$\Phi = \int \vec{E} \cdot d\vec{A}.$$

As one moves along the surface the direction of $\hat{n}$ varies. At every position there are two possible unit normals, differing by a sign. One can continuously move a unit normal around some surface. If doing this traces out a consistent family of normals then that family of normals is called an orientation and the surface is called orientable. For an orientable surface there are two possible choices of orientation. As an esoteric mathematical point, it should be mentioned that not all surfaces allow a choice of orientation. These surfaces are called nonorientable; a Mobius strip is an example. Nonorientable surfaces are not important for our purposes and we can always define the flux through a nonorientable surface to be zero.

Gauss's Law

Gauss's law relates the flux through a closed surface, called a Gaussian surface, to the total charge enclosed by the surface; they are proportional. For a closed surface we can eliminate the sign ambiguity in the flux by choosing the orientation associated with the outward normals.

Point Charge at the Center of a Sphere

We will develop Gauss's law gradually, first considering the case of a point charge at the center of a spherical Gaussian surface.

$$\Phi = \oint \vec{E} \cdot d\vec{A} = k_e\, Q \oint \frac{\hat{r}}{r^2} \cdot \hat{n}\, dA$$

A circle in the integral sign indicates that the integral is over a closed surface. Here, the integral is over a sphere of radius R. $\hat{n}$ is the unit normal to the sphere, but that is the same as the unit radial vector $\hat{r}$.

$$r = R = \text{constant} \quad \text{and} \quad \hat{r} = \hat{n} \implies \hat{r} \cdot \hat{n} = \hat{r} \cdot \hat{r} = 1$$

$$\Phi = k_e\, Q \oint \frac{\hat{r}}{r^2} \cdot \hat{n}\, dA = \frac{k_e\, Q}{R^2} \oint dA = \frac{k_e\, Q}{R^2} \times 4\pi R^2$$

$$= \ 4\pi\, k_e\, Q = \frac{Q}{\varepsilon_0}$$

Note that this is independent of the radius of the sphere; this must be the case because any field line that passes through one sphere will pass through another.

Point Charge with Any Closed Surface and Solid Angle

Considering any closed surface containing the charge, we can extend the previous result. Recall that flux is a measure of the number of field lines passing through a surface. A field line that passes through a sphere will also pass through any closed surface containing the charge. Note that when a field line enters a surface there is a negative contribution to the flux and a positive contribution when it leaves. A line that both enters and leaves has no net effect on the flux. It follows that any closed surface that contains the charge gets a flux Q/ε_0. Moreover any closed surface that does not contain the charge get a flux of zero.

The preceding discussion is less precise than one might think. We have appealed to the intuitive notion that flux is a measure of the number of field lines passing through a surface, but the discussion lacked precision. We may make these comments more precise by considering *solid angle*. An angle (in radians) can be viewed as an arc length of a unit circle; a complete unit circle having a length of 2π. A solid angle (in steradians) can similarly be defined as an area of a region on a unit sphere. The total area of a unit sphere is 4π, so that is the maximum solid angle.

Consider some infinitesimal piece of some surface, $d\vec{A}$ to be at some distance r from the charge at the origin. Generally, the dot product of any vector with a unit vector gives the component of the vector in the direction of the unit vector. $\hat{r}$ is the unit radial vector pointing away from the origin. It follows that $\hat{r}\cdot d\vec{A}$ is the radial component of $d\vec{A}$. We can also interpret $\hat{r}\cdot d\vec{A}$ as the projection of the surface $d\vec{A}$ onto a sphere of radius r. Dividing this by r^2 gives the projection of the infinitesimal surface $d\vec{A} = \hat{n}\, dA$ onto a a unit sphere. This then becomes the infinitesimal *solid angle* $d\Omega$,

$$d\Omega = \frac{\hat{r}}{r^2}\cdot\hat{n}\, dA.$$

Note that $\hat{r}\cdot\hat{n}$ is positive when a field line leaves a closed surface and the contribution to the area is positive. When it is negative the solid angle contribution is negative. We can now more precisely verify that the total flux of a point particle's field through a closed surface $\Phi = k_e\, Q \oint \frac{\hat{r}}{r^2}\cdot\hat{n}\, dA$ gives the value $4\pi\, k_e\, Q = Q/\varepsilon_0$ when the charge is inside the surface and gives 0 when it is outside.

Gauss's Law

If we now consider all the charges in the universe: $Q_1,\ Q_2,\ Q_3,\ \ldots$ and their electric fields $\vec{E}_1,\ \vec{E}_2,\ \vec{E}_3,\ \ldots$. The total electric field is then a sum

$$\vec{E} = \sum_i \vec{E}_i$$

and since the integral of a sum is the sum of the integrals we can write the total flux as a sum

$$\oint \vec{E}\cdot d\vec{A} = \sum_i \oint \vec{E}_i\cdot d\vec{A}$$

If Q_i is inside the surface then $\oint \vec{E}_i\cdot d\vec{A} = Q_i/\varepsilon_0$; if it is outside the integral gives zero. It follows that if Q_{inside} is the sum of all charge inside the surface, then we have Gauss's law.

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{\text{inside}}}{\varepsilon_0}.$$

Mathematically, Gauss's Law is equivalent to Coulomb's law in the case of electrostatics, where no charges are allowed to move. Here, Coulomb's law refers generally to the inverse square law for electric fields and forces. The equivalence means that one implies the other. We have just seen, at least loosely, that Coulomb's law implies Gauss's law. The reverse will be shown below, using spherical symmetry. Despite this mathematical equivalence, at the level of this course we will only be able to use find electric fields from Gauss's law in cases of symmetry.

Finding Electric Fields in Cases with Symmetry

The general steps for using symmetry to find the field at some point P in cases of symmetry follow. They are somewhat vague, but will be clarified by further examples.

Identify the symmetry and its effect on $\vec{E}$.

A symmetry in the charge distribution affects the possible form of the electric field. The three types of symmetry we will discuss are: spherical, cylindrical and planar.

Choose a Gaussian surface that reflects the symmetry and passes through the point P. $\oint \vec{E} \cdot d\vec{A} = E \times \text{(some area)}$

If we have spherical, cylindrical or planar symmetry then a spherical, cylindrical or planar surface will be perpendicular to the field and the magnitude of the field will be uniform over the surface. The flux through that surface will then be $E\,A$. In the cylindrical and planar cases these surfaces will not be closed surfaces, so they cannot be Gaussian surfaces. We can make these surfaces closed by adding pieces that are parallel to the field and then have on contribution to the flux.

Find the charge inside the Gaussian surface.

Q_{inside} is the total charge enclosed by the Gaussian surface. Often one must consider multiple cases.

Gauss's Law then gives $\vec{E}$.

The expressions found by the procedure sketched above can be plugged into Gauss's law to give the desired expression for the electric field.

Spherical Symmetry

Field of a Point Charge

The charge distribution of a point charge is spherically symmetric. If the charge position is taken as the origin then the charge distribution is the same under any rotation. This places important restrictions on the form of the field. The field at some point P must be in the radial direction; this follows from the fact that the field must be unchanged under some rotation about the line connecting the charge at the origin and P. Also, the field's magnitude must only vary with r, the distance from the origin to P. The general form of the field is:

$$\vec{E} = E(r)\,\hat{r}.$$

We need to evaluate the integral over a spherical Gaussian surface passing through P, which is a sphere of radius r. The unit normal vector is the same as the unit radial vector, $\hat{r} = \hat{n}$, which implies that $\hat{r} \cdot \hat{n} = \hat{r} \cdot \hat{r} = 1$. $E(r)$ is constant on the sphere and $\oint dA$ is just the surface area of the sphere.

$$\oint \vec{E} \cdot d\vec{A} = \oint E(r)\,\hat{r} \cdot \hat{n}\,dA = E(r) \oint dA = E(r)\,4\,\pi\,r^2$$

The only charge inside the Gaussian surface is the point charge, $Q_{\text{inside}} = Q$, so

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{inside}}}{\varepsilon_0} \implies \vec{E} = \frac{1}{4\,\pi\,\varepsilon_0}\,\frac{Q}{r^2}\,\hat{r} = k_e\,\frac{Q}{r^2}\,\hat{r}.$$

General Spherical Symmetry

The general problem of spherical symmetry follows by similar reasoning. The general form of the field is, by symmetry, the same as for the point charge $\vec{E} = E(r)\,\hat{r}$. The flux also follows similarly $\oint \vec{E}\cdot d\vec{A} = E(r)\,4\,\pi\,r^2$ and the electric field becomes

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{\text{inside}}}{\varepsilon_0} \implies \vec{E} = k_e\,\frac{Q_{\text{inside}}}{r^2}\,\hat{r}.$$

The value of Q_{inside} varies on a case-by-case basis.

For a uniform spherical shell of radius R and charge Q we have two cases:

$$r < R \implies Q_{\text{inside}} = 0 \implies \vec{E} = \vec{0}$$

$$r > R \implies Q_{\text{inside}} = Q \implies \vec{E} = k_e\,\frac{Q}{r^2}\,\hat{r}.$$

For a uniform solid sphere we have

$$r < R \implies Q_{\text{inside}} = Q\,\frac{r^3}{R^3} \implies \vec{E} = k_e\,\frac{Q}{R^3}\,r\,\hat{r}$$

$$r > R \implies Q_{\text{inside}} = Q \implies \vec{E} = k_e\,\frac{Q}{r^2}\,\hat{r}.$$

Cylindrical Symmetry

Cylindrical symmetry is the case where there is a rotational symmetry about an axis and a translational symmetry along the same axis. Cylindrical symmetry must be infinite. To specify the charge distribution we must use a charge density instead of giving a charge, because a finite charge spread over an infinite volume gives zero charge in any finite region. The field must point radially away from the symmetry axis. The field has the general form

$$\vec{E} = E(r)\,\hat{r},$$

which looks the same as the spherical result but is different. $\hat{r}$ is the unit vector away from the axis and r is the distance from it. The cylindrical surface at radius r, which I will refer to as a tube, will then be perpendicular to the field. This tube cannot be a Gaussian surface in itself because it isn't closed. We can make it a closed surface by adding flat ends to the tube. The field is parallel to the ends and the flux through them is zero. Take the length of the tube to be ℓ.

$$\oint \vec{E}\cdot d\vec{A} = \int_{\text{tube}} \vec{E}\cdot d\vec{A} + \int_{\text{end 1}} \vec{E}\cdot d\vec{A} + \int_{\text{end 2}} \vec{E}\cdot d\vec{A}$$
$$= \; E(r)\,A_{\text{tube}} + 0 + 0 = E(r)\,2\,\pi\,r\,\ell$$

This gives a general expression of the field with cylindrical symmetry as

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{\text{inside}}}{\varepsilon_0} \implies \vec{E} = \frac{1}{2\,\pi\,\varepsilon_0}\,\frac{Q_{\text{inside}}/\ell}{r}\,\hat{r}.$$

An example is an infinite uniform line of charge with linear density (charge/length) λ we get

$$Q_{\text{inside}} = \lambda\,\ell \implies \vec{E} = \frac{1}{2\,\pi\,\varepsilon_0}\,\frac{\lambda}{r}\,\hat{r}.$$

Planar Symmetry

Here we will consider just one example. Consider an infinite uniform plane of charge with surface charge density (charge/area) σ. The symmetry requires that the field is perpendicular to the plane. Drawing the field lines moving away from the line shows that the magnitude of the field must be uniform and thus independent of distance from the plane. If $\hat{n}$ is the unit vector pointing away form the plane then we get the field having the form

$$\vec{E} = E\,\hat{n}.$$

Take the Gaussian surface to be a right cylinder with faces of any cross sectional shape and area A with the faces on either side of the plane. The flat faces contribute a flux of $E\,A$ and the tube contributes zero flux. The charge inside the surface is $Q_{\text{inside}} = \sigma\,A$

$$\oint \vec{E}\cdot d\vec{A} = \int_{\text{tube}} \vec{E}\cdot d\vec{A} + \int_{\text{end 1}} \vec{E}\cdot d\vec{A} + \int_{\text{end 2}} \vec{E}\cdot d\vec{A}$$

$$= \; 0 + E\,A + E\,A = E\,2\,A$$

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{\text{inside}}}{\varepsilon_0} \implies \vec{E} = \frac{\sigma}{2\,\varepsilon_0}\,\hat{n}.$$

Conductors and Insulators

Inside a conductor there are freely moving charges. Inside an insulator there are not. For typical conductors, metals, the charge carriers are electrons; most of the electron are tied to their atoms but a small number of electrons from each atoms are shared by all atoms in a sea of electrons, which are free to move. In semiconductors, the conduction mechanism is different. The charge carriers are either electrons jumping between atomic sites or positively charged holes (A hole is lack of an electron.) jumping between atoms; they are called n-type and p-type semiconductors for negative (electrons) and positive (holes) charge carriers. In an insulator all electrons are either tied to their atom or to bonds between atoms and are not free to move.

Conductors in Electrostatics - I

Inside a conductor $\vec{E} = \vec{0}$.

If there is an electric field inside a conductor any free charges will move. This means that an electric field inside a conductor implies a current. Since currents are not allowed in electrostatics, it follows that the field inside a conductor must be zero.

There is no excess charge inside a conductor. All excess charge is on the surface of a conductor.

Consider a Gaussian surface entirely inside a conductor. Since the electric field is zero, Gauss's law implies that $Q_{\text{inside}} = 0$. This means there is no excess charge in *any* region inside a conductor. There can be excess charge on a conductor, though. All excess charge is on the surface.

The electric field is perpendicular to the surface of a conductor and it is proportional to the surface charge density, $E = \sigma / \varepsilon_0$.

For the same reason that the field is zero inside a conductor, it must be perpendicular to the surface of a conductor. If there is a component parallel to the surface of a conductor then that will induce surface currents and this violates the assumptions of electrostatics.

Consider a small cylindrical Gaussian surface with small flat faces parallel to the surface of the conductor, one surface inside and one outside. Write this as a tube and two ends as before. The tube is parallel to the field wo the flux through it is zero. The field is zero inside the conductor, so the inside end gives zero flux. This leaves a flux of $E\,A$ at the outside end.

$$\oint \vec{E}\cdot d\vec{A} = \int_{\text{tube}} \vec{E}\cdot d\vec{A} + \int_{\text{inside end}} \vec{E}\cdot d\vec{A} + \int_{\text{outside end}} \vec{E}\cdot d\vec{A} = 0 + 0 + E\,A = E\,A$$

If the surface charge density at that poistion is σ then the charge inside the Gaussian surface is $Q_{\text{inside}} = \sigma\,A$. If we write $\hat{n}$ as the outward unit normal at the surface of the conductor then Gauss's law gives the surface field as

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\,\hat{n}.$$

Conductors and the Definition of $\vec{E}$.

A point charge is always attracted to a neutral conductor. Consider a positive point charge Q and a neutral conductor. The positive charge will move charge in the conductor; near Q there will be a negative charge buildup and the far end of the conductor will have a positive charge. Although there is as much positive as negative in the conductor, the negative is closer and will have a larger effect. This gives a net attractive force.

If we revisit the definition of the electric field, we see a problem. Let q_0 be some test charge. The effect of the test charge is to move around charge in conductors. This means that the presence of the test charge causes a change in the field and $\vec{E} = \vec{F}/q_0$ depends on q_0. If we want to keep this definition of the field then we must insist that all charge is kept fixed in place, but with conductors this will not happen. For instance, moving a test charge near a neutral conductor will create a field and thus the definition will give a field where there is none.

We will modify the definition that noting that the force between a neutral conductor and charge is proportional to the q_0^2, so we can modify the definition by taking the limit

$$\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}}{q_0}.$$

This gives the correct field in the presence of conductors.