

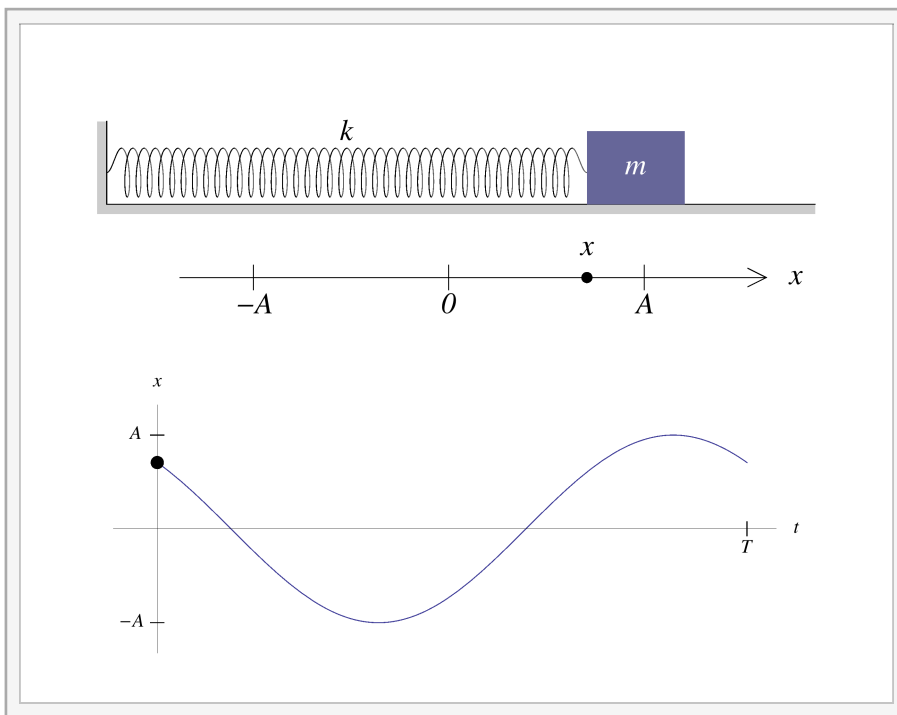
Chapter J

Oscillatory Motion

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Simple Harmonic Motion

The Mass-Spring System



Interactive Figure

Consider a mass sliding without friction on a horizontal surface. The force of a spring is given by Hooke's law $F = -kx$. Applying Newton's second law gives a second order ODE

$$F_{\text{net}} = ma \implies -kx = m \frac{d^2x}{dt^2}$$

Defining the angular frequency by

$$\omega = \sqrt{\frac{k}{m}},$$

gives the second order ODE

$$0 = \frac{d^2 x}{dt^2} + \omega^2 x.$$

The Differential Equation for Simple Harmonic Motion

We need to distinguish our standard example of simple harmonic motion, the mass-spring system, from the general phenomenon. Generally simple harmonic motion occurs any time a mechanical system gives rise to a differential equation of the form

$$0 = \frac{d^2 x}{dt^2} + \omega^2 x.$$

We will see that this is quite a generic expression.

Comments on ODEs (Ordinary Differential Equations)

A differential equation (DE) is some equation involving a function and its derivatives.

The differential equation is solved to find the function.

The order of a DE is the highest number of derivatives.

If there is at most a second derivative it is a second order equation.

If it is a function of one variable it is an ordinary differential equation (ODE). For functions of several variables there are partial differential equations (PDE).

For functions of more than one variable we take partial derivatives instead of ordinary ones. The differential equations course (taken after Cal. III) is on ODEs.

The general solution of a p^{th} order ODE is any solution involving p independent arbitrary constants.

It is easy to verify that something is a solution to a differential equation; it is just a matter of taking derivatives and plugging into an equation. If a solution has the correct number of arbitrary constants then we can conclude that this is the general solution.

The General Solution of the Differential Equation

With the analogy clearly stated, let us solve the second order ordinary differential equation

$$0 = \frac{d^2 x}{dt^2} + \omega^2 x.$$

Since the second derivatives of both sine and cosine are the negatives of themselves, it follows that

$$\cos \omega t \text{ and } \sin \omega t$$

are solutions to our differential equation. Because the ODE has the simple properties (a homogeneous linear equation) that:

- (i) a constant times a solution is a solution and
- (ii) the sum of two solutions is a solution,

it follows that

$$x(t) = B \cos \omega t + C \sin \omega t$$

is a solution, where B and C are arbitrary constants. Since we have solution to a second order ODE with two arbitrary constants, we can conclude that this is the general solution. We can interpret the constants B and C in terms of the initial position x_0 and the initial velocity v_0 .

$$x_0 = x(0) = B \cdot 1 + C \cdot 0 = B$$

$$v(t) = \frac{d}{dt} x(t) = -\omega B \sin \omega t + \omega C \cos \omega t$$

$$\Rightarrow v_0 = v(0) = -\omega B \cdot 0 + \omega C \cdot 1 \Rightarrow C = \frac{v_0}{\omega}$$

This gives

$$x(t) = x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t.$$

Another way of presenting this solution is

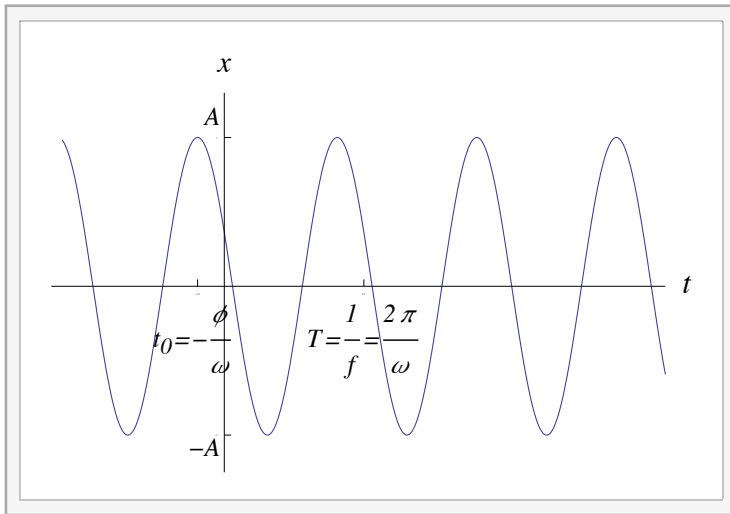
$$x(t) = A \cos(\omega t + \phi)$$

where the arbitrary constants are A , the amplitude, and ϕ , the phase angle. ω is called the angular frequency. This is related to the period T and the frequency f . The period is the time for one cycle. After one period the argument of the trig function shifts by 2π .

$$t \rightarrow t + T \Rightarrow (\omega t + \phi) \rightarrow (\omega t + \phi) + 2\pi$$

This gives $\omega T = 2\pi$. The frequency is the number of cycles per time; since the time per cycle is the period we get $f = 1/T$. Combining this gives

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \quad \text{and} \quad f = \frac{1}{T} = \frac{\omega}{2\pi}.$$



Interactive Figure

Energy Considerations

The Energy of a Mass-Spring System

The total energy is the sum of kinetic and potential energies. When the mass is at the turning points $x = \pm A$, its speed is zero; we can then write $E = (1/2)kA^2$. When it passes the equilibrium point $x = 0$ the potential energy is zero and thus its kinetic energy is the maximum; so must the speed be its maximum, $v = \pm v_{\max}$. This allows us to write the energy with two equivalent forms for the total energy.

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \begin{cases} \frac{1}{2}kA^2 \\ \frac{1}{2}mv_{\max}^2 \end{cases}$$

Equating the two expressions for the total energy gives an expression for the maximum speed in terms of the amplitude.

$$v_{\max} = \sqrt{\frac{k}{m}} A = \omega A$$

Speed and Position for General Simple Harmonic Motion

For general simple harmonic motion we have

$$x(t) = A \cos(\omega t + \phi)$$

Taking the time derivative of this gives the velocity.

$$v(t) = -\omega A \sin(\omega t + \phi)$$

Since both sin and cos vary between ± 1 we can identify the maximum speed as

$$v_{\max} = \omega A.$$

This is equivalent to what we had for the mass-spring case. The point here is to show that this is generally true. If the mass-spring energy is written in terms of the amplitude A then we can solve for v and, using the mass-spring value of ω , get

$$v = \pm \omega \sqrt{A^2 - x^2}.$$

This expression is also generally true for simple harmonic motion. To verify that generally, we can write $\cos(\omega t + \phi) = x/A$ and $\sin(\omega t + \phi) = -v/(\omega A)$. Using $\cos^2 + \sin^2 = 1$ we can get the result.

The Vertical Mass-Spring

When a mass hangs from a spring we need to add the effect of gravity. As before, the second law gives our differential equation.

$$F_{\text{net}} = ma \implies -kx + mg = m \frac{d^2 x}{dt^2}$$

The equilibrium position of this is when the forces cancel $F_{\text{net}} = 0$. This gives

$$k x_{\text{eq}} = m g.$$

We can redefine our coordinates relative to the new equilibrium position.

$$x = x' + x_{\text{eq}}$$

If we insert this into our differential equation we get

$$-k x' = m \frac{d^2 x'}{dt^2}.$$

Here we have used the value of x_{eq} and the fact that the derivatives of constants vanish, which implies the second derivative of x is the same as the second derivative of x' .

The interpretation of the above expression is simple. The effect of gravity is trivial. It just shifts the equilibrium position and we end up with simple harmonic motion about the new equilibrium position.

The Physical and Simple Pendulums

The Physical Pendulum

Consider a rigid body rotating without friction about an axis. The center of mass is a distance d from the axis. At equilibrium, the center of mass will hang below the center. Take the angle θ to be the angle of the line from the axis to the center of mass measured from vertical; note that $\theta = 0$ is the equilibrium position. The only nonzero torque acting on the rigid body is the torque due to gravity. This is

$$\tau_{\text{net}} = \tau_{\text{grav}} = -m g d \sin \theta.$$

The direction of positive θ gives our sign convention for torque. The reason for the minus sign in the above expression is the torque tends toward smaller angles. Since the angular acceleration is the second time derivative of the angle, the rotational second law gives a differential equation.

$$\tau_{\text{net}} = I \alpha \implies -m g d \sin \theta = I \frac{d^2 \theta}{dt^2}$$

$$\frac{d^2 \theta}{dt^2} = -\frac{m g d}{I} \sin \theta$$

If we define

$$\omega = \sqrt{\frac{m g d}{I}}$$

then we get a differential equation of the form

$$\frac{d^2 \theta}{dt^2} = -\omega^2 \sin \theta.$$

This is almost of the form of our simple harmonic motion equation $d^2 x / dt^2 = -\omega^2 x$, except for the sin function. If we consider small angles then we get

$$\sin \theta \approx \theta \text{ for small } \theta.$$

We can then conclude that for small amplitude oscillations we have simple harmonic motion with an angular frequency ω given by the expression above.

The Simple Pendulum

The simple pendulum is a special case of the physical pendulum. It is the case where all the mass m is located at a point, the pendulum bob. If the bob is on the end of a string of length L then we get

$$d = L \text{ and } I = mL^2.$$

Solving for ω we get

$$\omega = \sqrt{\frac{g}{L}}.$$