Chapter H

Rotational Motion and Equilibrium

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Kinematics of Rotations about a Fixed Axis

Rigid Bodies and Rotations in General

The distance between any two positions in a rigid body is fixed. A book can be viewed as a rigid body as long as it is kept closed; when it is opened then the distance between a point on the back cover and a point on the front cover varies and it is not rigid.

A rotation is described by an axis and an angle. An axis is a line. The axis of rotation of a door is its hinge. The axis of a tire is its axle. Often in a planar diagram we will draw an axis as a point. The axis is then the line perpendicular to that plane through the point. A rotation is about some axis and by some angle. Note that when a rigid body rotates different points move different distances. The distance a point moves s is proportional to the (perpendicular) distance from the axis r, but the ratio s/r is the same for any two points. This ratio is just the angle of rotation in radians.

In general, three dimensional rotations about different axes do no commute; this means that changing the order of two rotations gives a different answer. Try this with your book as the rigid body. Rotate the book by a 90° angle about the vertical axis and then by 90° about a horizontal axis. If the book is returned to its original position and the two rotations are repeated in the opposite order then the book ends up in a different orientation.

Kinematical Variables

To understand rotational kinematics it is essential to appreciate the analogy to one dimensional kinematics. Recall that a one dimensional vector is a real number and that its direction is given by its sign. The rotational analog of the position is the angle of rotation. The other kinematical variables follow:

	One Dimensional Linear Motion	**Rotations about a Fixed Axes**
Position	$x\ (m)$	θ (angle in rad)
Velocity	$v\ (m/s)$	ω (angular velocity in rad$/s$)
Average	$\overline{v} = \frac{\Delta x}{\Delta t}$	$\overline{\omega} = \frac{\Delta \theta}{\Delta t}$
Instantaneous	$v = \frac{dx}{dt}$	$\omega = \frac{d\theta}{dt}$
Acceleration	$a\ \left(m/s^2\right)$	α (angular acceleration in rad$/s^2$)
Average	$\overline{a} = \frac{\Delta v}{\Delta t}$	$\overline{\alpha} = \frac{\Delta \omega}{\Delta t}$
Instantaneous	$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$

Constant Angular Acceleration

Since the rotational variables θ, ω and α are interrelated the same as x, v and a, we can find expressions for constant angular acceleration.

One Dimensional Linear Motion	**Rotations about a Fixed Axes**
$v = v_0 + a\,t$	$\omega = \omega_0 + \alpha\,t$
$\Delta x = \frac{1}{2}\left(v_0 + v\right)\,t$	$\Delta\theta = \frac{1}{2}\left(\omega_0 + \omega\right)\,t$
$\Delta x = v_0\,t + \frac{1}{2}\,a\,t^2$	$\Delta\theta = \omega_0\,t + \frac{1}{2}\,\alpha\,t^2$
$v^2 - v_0^2 = 2\,a\,\Delta x$	$\omega^2 - \omega_0^2 = 2\,\alpha\,\Delta\theta$

Relation Between Linear and Rotational Variables

In chapter E we described circular motion in terms of centripetal and tangential coordinates, where the centripetal direction is toward the center of the circle and the tangential direction is in the direction of motion. A point on a rigid body a distance r from the center moves in a circle of radius r, so that discussion is relevant here.

The velocity is purely tangential

$$\vec{v} = v_t\,\hat{u}_t.$$

v_t is just the speed, which is just $v_t = \frac{ds}{dt}$ where ds is the infinitesimal arc length. Since by the definition of angles in radians with is related to the infinitesimal angle $ds = r\,d\theta$. Since $d\theta/dt = \omega$ we get

$$\vec{v} = v_t\,\hat{u}_t = r\,\omega\,\hat{u}_t \text{ or in other words } v_t = r\,\omega \text{ and } v_c = 0.$$

In chapter E we saw the acceleration had the form

$$\begin{aligned}\vec{a} &= a_c\,\hat{u}_c + a_t\,\hat{u}_t \\ &= \frac{v^2}{r}\,\hat{u}_c + \frac{dv}{dt}\,\hat{u}_t\,.\end{aligned}$$

We can now rewrite the centripetal acceleration in terms of the rotational variables

$$a_c = \frac{v^2}{r} = \frac{(r\,\omega)^2}{r} = \omega^2\,r.$$

The tangential component of acceleration can be written similarly.

$$a_t = \frac{dv}{dt} = \frac{d\,r\,\omega}{dt} = r\,\frac{d\,\omega}{dt} = r\,\alpha.$$

Summarizing for the acceleration

$$\begin{aligned}\vec{a} &= a_c\,\hat{u}_c + a_t\,\hat{u}_t \\ &= \omega^2\,r\,\hat{u}_c + r\,\alpha\,\hat{u}_t\,.\end{aligned}$$

Dynamics of Rigid Bodies Rotating about an Axis

Summary and Analogy with One Dimensional Motion

	One Dimensional Linear Motion	**Rotations about a Fixed Axes**
Kinematics	$x,\ v,\ a$	$\theta,\ \omega,\ \alpha$
Force	F	τ (torque)
Inertia	m	I (moment of inertia)
Momentum	$p = m\,v$	$L = I\,\omega$ (angular momentum)
Second Law	$F_{\text{net}} = m\,a$ $F_{\text{net}} = \frac{d}{dt}p$	$\tau_{\text{net}} = I\,\alpha$ $\tau_{\text{net}} = \frac{d}{dt}L$
Conservation of Momentum	$F_{\text{net}}^{\text{ext}} = 0$ $\Longrightarrow \Delta\,p_{\text{tot}} = 0$	$\tau_{\text{net}}^{\text{ext}} = 0$ $\Longrightarrow \Delta\,L_{\text{tot}} = 0$
Kinetic Energy	$K = \frac{1}{2}\,m\,v^2$	$K = \frac{1}{2}\,I\,\omega^2$
Work	$W = \int F\,dx$	$W = \int \tau\,d\theta$
Work-Energy Theorem	$W_{\text{net}} = \Delta\,K$	$W_{\text{net}} = \Delta\,K$
Power	$\mathcal{P} = \frac{dW}{dt} = F\,v$	$\mathcal{P} = \frac{dW}{dt} = \tau\,\omega$

This table is an extension of the preceding tables for kinematics. Now we consider dynamics. Dynamical quantities are things like force and mass. The rotational analog of force is called torque and the rotational analog of mass is the moment of inertia. These two quantities are undefined in the table; their definitions follow. For all the other quantities, the above table serves as the definitions of the variables.

Kinetic Energy and the Definition of the Moment of Inertia

Consider a rigid body consisting of point masses m_i. The perpendicular distance from the axis to the i^{th} mass is r_i. If the rigid body rotates with angular velocity ω then the speed of the i^{th} mass is

$$v_i = r_i\,\omega.$$

The total kinetic energy is the sum of the kinetic energies of all the masses. Using the above expression for the speed we get

$$K = \frac{1}{2}\sum_i m_i\,v_i^2 = \frac{1}{2}\sum_i m_i\,r_i^2\,\omega^2 = \frac{1}{2}\left(\sum_i m_i\,r_i^2\right)\omega^2.$$

Using our desired expression for the kinetic energy $K = \frac{1}{2}\,I\,\omega^2$ we get the expression for moment of inertia for a rigid body about some axis.

$$I = \sum_i m_i\,r_i^2.$$

This expression is for a discrete distribution; this means that the distribution is a collection of point masses.

For a continuous distribution we replace the sum with an integral. Break up the distribution into in infinite number of infinitesimal pieces. Take dm to be the mass of one of the infinitesimal pieces. The perpendicular distance from the axis to dm is r. The total mass is M which is just the sum (integral) of all the infinitesimal pieces.

$$M = \int dm$$

The moment of inertia is the sum of all the $r^2\, dm$. This gives

$$I = \int r^2\, dm.$$

Moments of Inertia for Uniform Bodies

We can use the formula above to calculate the moments of inertia for certain simple geometric shapes with uniform mass distributions. By uniform distribution of mass we mean the density is constant throughout the body.

Thin Rod about Perpendicular Axis Through End

Consider a rod of negligible thickness, mass M and length L. Take the x axis to be along the rod with $x = 0$ at the axis. To integrate over the whole rod we will choose x as our integration variable. The limits of integration are

$$0 \le x \le L.$$

dm is the mass between x and $x + dx$. Since the distribution is uniform we can conclude that the fraction of the mass is the same as the fraction of the length. The fraction of the length is dx/L. This gives:

$$dm = \frac{M}{L}\, dx.$$

The perpendicular distance from the axis to x is $r = |x|$.

Putting all this together we can write the moment as an integral.

$$I = \int r^2\, dm = \int_0^L x^2 \frac{M}{L}\, dx = \frac{M}{L}\left(\frac{1}{3} L^3 - 0\right)$$

This gives our result

$$I = \frac{1}{3} M L^2.$$

Thin Rod about Perpendicular Axis Through Center

This is the same as before except that our limits of integration are different. The limits of integration are

$$-\frac{L}{2} \le x \le \frac{L}{2}.$$

The expression for dm is the same and the integral becomes:

$$I = \int r^2\, dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L}\, dx = \frac{M}{L}\,\frac{1}{3}\left(\frac{L^3}{8} - -\frac{L^3}{8}\right).$$

We then get

$$I = \frac{1}{12} M L^2.$$

Hoop or Thin-shelled Hollow Cylinder about Perpendicular Axis through Center

First consider a hoop of mass M and radius R rotating about a perpendicular axis through the center. r is the distance from the axis to the infinitesimal mass dm. All the mass is at the same distance

$$r = R = \text{constant}.$$

It is possible to find I without actually performing an integral.

$$I = \int r^2\, dm = \int R^2\, dm = R^2 \int dm.$$

Since $M = \int dm$ we get

$$I = M R^2.$$

Now consider a thin-shelled hollow cylinder about the central axis. It is still true that all the mass is the same perpendicular distance of $r = R$ from the axis and the above formula still applies.

Disk or Solid Cylinder about Perpendicular Axis through the Center

It should now be clear that the moment for a disk should be the same as a solid cylinder. We can break up a disk into concentric thin rings of radius r with thickness dr. The limits of integration become

$$0 \le r \le R.$$

The infinitesimal area of a thin ring can be written as the length of the ring, which is the circumference $2\pi r$ multiplied by its thickness dr.

$$dA = 2\pi r\, dr$$

The uniform distribution implies that

$$dm = \frac{M}{A_{\text{tot}}}\, dA = \frac{M}{\pi R^2}\, 2\pi r\, dr \implies dm = \frac{2M}{R^2}\, r\, dr.$$

$$I = \int r^2\, dm = \int_0^R r^2\, \frac{2M}{R^2}\, r\, dr = \frac{2M}{R^2} \int_0^R r^3\, dr = \frac{2M}{R^2}\left(\frac{1}{4} R^4 - 0\right).$$

The final result is

$$I = \frac{1}{2} M R^2.$$

Thin-shelled Hollow Sphere about Axis through Center

The integral for the moment is $I = \int r^2\, dm$, where r is the distance from an axis. The r used in three dimensions is not the distance from an axis but the distance from an origin $\sqrt{x^2 + y^2 + z^2}$. To avoid confusion between the different r we will refer to this last value as ρ.

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

For *any* spherical distribution our calculation is made awkward because the integral involves r, but the distribution is symmetric with respect to ρ. It is possible to perform the necessary integrals but here we will introduce a trick that will greatly simplify the integration. The perpendicular distance from the z axis is $\sqrt{x^2 + y^2}$, so the moment of inertia about the z axis is

$$I_z = \int r^2\, dm = \int \left(x^2 + y^2\right) dm.$$

We can similarly find the moments about the x and y axes.

$$I_x = \int r^2\, dm = \int \left(y^2 + z^2\right) dm$$

$$I_y = \int r^2\, dm = \int \left(x^2 + z^2\right) dm$$

Since we have spherical symmetry these three moments must be the same $I_x = I_y = I_z$. It follows that we can write the moment as 1/3 the sum of the three. Doing this we get an integral in terms of ρ instead of r.

$$I = \frac{1}{3}\left(I_x + I_y + I_z\right) = \frac{2}{3} \int \left(x^2 + y^2 + z^2\right) dm = \frac{2}{3} \int \rho^2\, dm$$

Applying this general spherical trick to the case of a thin-shelled hollow sphere the resulting integral is trivial. All the mass in this case is at the same distance, the radius of the sphere $\rho = R =$ constant.

$$I = \frac{2}{3} \int \rho^2\, dm = \frac{2}{3} \int R^2\, dm = \frac{2}{3} R^2 \int dm$$

The final result becomes

$$I = \frac{2}{3} M R^2$$

Solid Sphere about Axis through Center

Now we consider a uniform solid sphere. We can use the general spherical trick discussed above in this case as well. Here we break up the sphere into thin concentric spheres. Take the integration variable to be ρ and its limits to be

$$0 \le \rho \le R.$$

The infinitesimal volume of a thin sphere can be written as the area of the sphere, which is the area $4\pi\rho^2$ multiplied by its thickness $d\rho$.

$$dV = 4\pi\rho^2\, d\rho$$

The uniform distribution implies that

$$dm = \frac{M}{V_{\text{tot}}} dV = \frac{M}{\frac{4}{3}\pi R^3} 4\pi\rho^2\, d\rho \implies dm = \frac{3M}{R^3}\rho^2\, d\rho.$$

Inserting this into our integral we get

$$I = \frac{2}{3} \int \rho^2\, dm = \frac{2}{3} \int_0^R \rho^2 \frac{3M}{R^3} \rho^2\, d\rho = \frac{2M}{R^3} \int_0^R \rho^4\, d\rho.$$

The final result becomes

$$I = \frac{2}{5} M R^2.$$

Perpendicular-Axis Theorem

I_x, I_y and I_z are the moments of inertia about the x, y and z axes. For any planar (flat) object in the xy-plane

$$I_z = I_x + I_y.$$

The proof is simple. If the object is entirely in the xy-plane then $z = 0$ for the entire mass distribution. Since r is the perpendicular distance form the axis in the definition $I = \int r^2\, dm$, we have

$$I_x = \int \left(y^2 + z^2\right) dm = \int y^2\, dm \text{ and } I_y = \int \left(x^2 + z^2\right) dm = \int x^2\, dm.$$

The proof follows easily.

$$I_z = \int \left(x^2 + y^2\right) dm = \int x^2\, dm + \int y^2\, dm = I_x + I_y$$

As an example, consider rectangular plate about perpendicular axis through the center, Take the plate to have dimensions $a \times b$ and take the a to be the length in the x direction and b to be the y length. I_x and I_y are equivalent to the moments of uniform rods about the center or length b and a.

$$I = I_z = I_x + I_y = \frac{1}{12} M\, b^2 + \frac{1}{12} M\, a^2 \implies I = \frac{1}{12} M \left(a^2 + b^2\right).$$

Parallel-Axis Theorem

The parallel-axis theorem relates the moment of a rigid body about some axis to the moment about the axis parallel to the first and passing through the center of mass. If I is the moment about an axis, I_{cm} is the moment about the parallel axis that passes through the center of mass (the center of mass axis) and D is the distance between the two axes, then the parallel axis theorem is

$$I = I_{\text{cm}} + M\, D^2.$$

To prove this take $\vec{r}$ to be the perpendicular vector from the original axis to the infinitesimal mass dm. Take $\vec{r}'$ to be the perpendicular vector from the center of mass axis to dm. Take $\vec{D}$ as the perpendicular vector from the original axis to the center of mass axis. These vectors are related by

$$\vec{r} = \vec{r}' + \vec{D}.$$

Squaring this gives

$$r^2 = \left(\vec{r}' + \vec{D}\right)\cdot\left(\vec{r}' + \vec{D}\right) = r'^2 + D^2 + 2\,\vec{D}\cdot\vec{r}'.$$

With this we can rewrite the moment of inertia

$$I = \int r^2\, dm = \int r'^2\, dm + D^2 \int dm + 2\,\vec{D}\cdot \int \vec{r}'\, dm.$$

The first term is just I_{cm} the second is $M\, D^2$ and the third must be zero for our result to be true. $\int \vec{r}'\, dm$ is related to the center of mass in the $\vec{r}'$ coordinates and this is just zero $\int \vec{r}'\, dm = M\, \vec{r}'_{\text{cm}} = \vec{0}$

Energy and Rigid Bodies

Gravitational Potential Energy

It is a straightforward matter to find the potential energy of a rigid body.

$$U = \sum_i m_i\, g\, y_i = g \sum_i m_i\, y_i = g\, M\, y_{\text{cm}}$$

Here M is the total mass and y_{cm} is the height of the center of mass. It follows that the total potential of a rigid body is

$$U = M\, g\, y_{\text{cm}}.$$

This is easy to interpret. When calculating the potential energy of a rigid body we treat the body as if all the mass is at the center of mass.

Rotation with Translation - Rolling Motion

Now consider the case of a body with both rotational and translational motion. As examples think of a spinning ball and of a rolling object. For a system of particles we derived the result

$$K_{\text{tot}} = K'_{\text{tot,cm}} + \frac{1}{2} M\, v_{\text{cm}}^2.$$

K_{tot} is the total kinetic energy of the system and $K'_{\text{tot,cm}}$ is the total kinetic energy in the center of mass frame. This center of mass energy is, for a rotating rigid body, just

$$K'_{\text{tot,cm}} = \frac{1}{2} I_{\text{cm}}\, \omega^2.$$

It follows that we can write

$$K_{\text{tot}} = \frac{1}{2} M\, v_{\text{cm}}^2 + \frac{1}{2} I_{\text{cm}}\, \omega^2.$$

In the case of a rolling body we have a rolling constraint that a body rolls without slipping. This is that the arc length along the rolling radius of the body is the same as the distance Δx it moves along the surface it rolls on. If it rotates by an angle $\Delta\theta$ then the arc length is $R\,\Delta\theta$. The constraint becomes

$$R\,\Delta\theta = \Delta x.$$

Since the velocity is $v = dx/dt$ and the angular velocity is $\omega = d\theta/dt$ the rolling constraint becomes

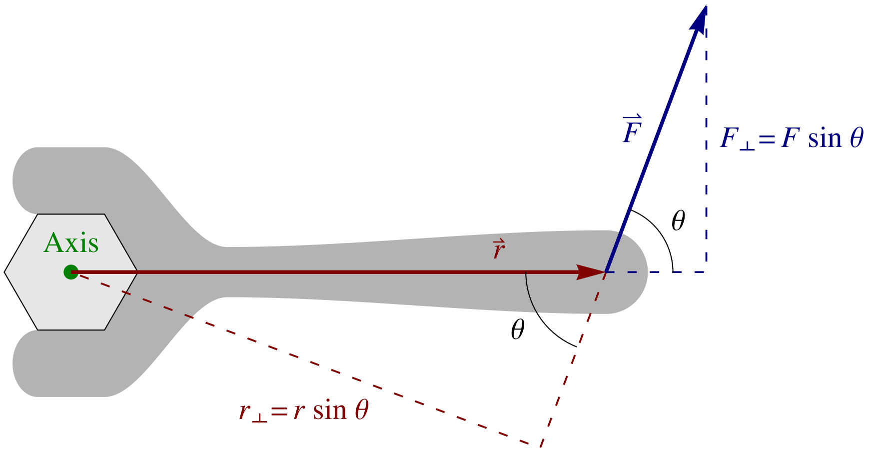
$$R\,\omega = v.$$

Taking another derivative gives the acceleration and angular acceleration and we get

$$R\,\alpha = a.$$

The Vector Nature of Rotational Quantities

Torque about an Axis



We define torque as the rotational analog of force. Suppose you are trying to loosen a bolt. The axis of rotation is the center of the bolt. If you are unable to give sufficient torque with your hand you grab a wrench. Take $\vec{r}$ as the vector from the axis to where the force $\vec{F}$ is applied. Clearly the important part of the force is the component of the force perpendicular to the radial vector $\vec{r}$. Moreover the larger r is the larger the torque. This motivates the definition of torque

$$\tau = r\,F_\perp$$

If θ is the angle between $\vec{r}$ and $\vec{F}$ then we can write $F_\perp = F \sin\theta$. Similarly we can $r_\perp = r \sin\theta$ as the component of $\vec{r}$ perpendicular to $\vec{F}$. This gives us other ways of writing the torque.

$$\tau = r\,F_\perp = r\,F \sin\theta = r_\perp\,F$$

The sign of torque depends on the sign convention for kinematics. If a force tends to make something rotate in the positive direction then the torque is positive and similarly negative torques tend to make things rotate in the negative direction.

Angular Velocity and Torque as Vectors

A rotation has a magnitude, the angle of rotation, and a direction, along the direction of the axis. Rotations are not vectors, though. Vector addition is commutative but rotations are not. It turns out that infinitesimal rotations do commute and *are* vectors. We can write an infinitesimal rotation as $d\vec{\theta}$. Since angular velocity about an axis requires only an infinitesimal rotation, $\omega = d\theta/dt$, we can define the angular velocity vector

$$\vec{\omega} = \frac{d\vec{\theta}}{dt}\,.$$

There are two possible directions along an axis. We decide which direction by using the right hand rule. Wrap the fingers of your right hand in the direction of rotation. The thumb points in the direction of the vector. If angular velocity is a vector then we can also make angular acceleration a vector.

$$\vec{\alpha} = \frac{d}{dt}\,\vec{\omega}$$

Wit these considerations we can now make a vector out of the torque. We can assign its direction to the sense of rotation due to that torque. $\vec{r}$ and $\vec{F}$ are vectors; we will define the cross product so that the cross product of $\vec{r}$ and $\vec{F}$ is the torque $\vec{\tau}$.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

The Cross or Vector Product

The dot product, or scalar product, is a way of multiplying of two vectors that gives a scalar. The cross product, also known as the vector product, is a multiplication that gives a vector.

$$\vec{A} \times \vec{B} \text{ is } a \text{ vector.}$$

The magnitude of this vector is $A\,B \sin\theta$. We will specify the direction with a unit vector $\hat{u}$. The two vectors $\vec{A}$ and $\vec{B}$ define a plane; their cross product is perpendicular to that plane. There are two unit vectors perpendicular to any plane; we use the right hand rule to find the correct one. Put your right thumb in the direction of the first entry $\vec{A}$ and your fingers in the direction of the second entry $\vec{B}$. The palm of your hand is in the direction $\hat{u}$, giving the direction of the cross product.

$$\vec{A} \times \vec{B} = A\,B \sin\theta\ \hat{u}\ \ (\hat{u} \text{ by right hand rule})$$

Properties of the Cross Product

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}\ \ \text{(antisymmetry } - \text{ not commutative)}$$

$$\left(\vec{A} \times \vec{B}\right) \times \vec{C} \neq \vec{A} \times \left(\vec{B} \times \vec{C}\right)\ \ \text{(not associative)}$$

$$\left(c\,\vec{A}\right) \times \vec{B} = c\left(\vec{A} \times \vec{B}\right) = \vec{A} \times \left(c\,\vec{B}\right)\ \ (\text{associative } w.r.t. \text{ scalar mult.})$$

$$\left(\vec{A} + \vec{B}\right) \times \vec{C} = \vec{A} \times \vec{C} + \vec{B} \times \vec{C}\ \ \text{and}$$

$$\vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}\ \ \text{(distributive)}$$

The Cross Product and Components

With the dot product we were able to write it in terms of components. We can do the same for the cross product. As before we have to find the products of the basis unit vectors. Because of the antisymmetry property we get $\vec{A} \times \vec{A} = \vec{0}$. It follows then that

$$\vec{0} = \hat{x} \times \hat{x} = \hat{y} \times \hat{y} = \hat{z} \times \hat{z}.$$

Using our definition of the cross product we can see that the cross product of two perpendicular unit vectors is a third unit vector perpendicular to the two.

$$\vec{A} \times \vec{B} = A\,B \sin\theta\ \hat{u} \implies \hat{v} \times \hat{w} = 1 \cdot 1 \cdot 1\ \hat{u}$$

The cross product of the unit vectors $\hat{x}$ and $\hat{y}$ is thus a unit vector perpendicular to the xy plane. This is either $\hat{z}$ or $-\hat{z}$. We insist that our coordinate system is right-handed; this means that

$$\hat{x} \times \hat{y} = \hat{z}.$$

For the other combinations of unit vectors there is a simple rule to keep track of their cross products. Arrange x, y and z around a circle.

$$\begin{matrix} & x & \\ z & & y \end{matrix}$$

If the order of the three coordinates has the same sense of rotation as x, y, z it gains a positive sign. If opposite it gets a minus sign.

$\hat{y}\times\hat{z}=\hat{x}, \quad \hat{z}\times\hat{x}=\hat{y}\,,$
$\hat{y}\times\hat{x}=-\hat{z}, \quad \hat{x}\times\hat{z}=-\hat{y}$ and $\hat{z}\times\hat{y}=-\hat{x}$

We can put all this together and get the cross product in terms of components.

$$\begin{aligned}\vec{A}\times\vec{B} &= \left(A_x\,\hat{x}+A_y\,\hat{y}+A_z\,\hat{z}\right)\times\left(B_x\,\hat{x}+B_y\,\hat{y}+B_z\,\hat{z}\right)\\ &= \hat{x}\left(A_y\,B_z-A_z\,B_y\right)+\hat{y}\,(A_z\,B_x-A_x\,B_z)+\hat{z}\left(A_x\,B_y-A_y\,B_x\right)\end{aligned}$$

Angular Momentum of a Particle and Torque

We previously defined the torque as

$$\vec{\tau}=\vec{r}\times\vec{F}.$$

We can similarly define the angular momentum of a particle as

$$\vec{L}=\vec{r}\times\vec{p}.$$

Both of these expressions are relative to an origin; $\vec{r}$ is the position vector. It is from the origin to the position to where the force is applied in the case of torque. It is from the origin to the position of the particle for the angular momentum.

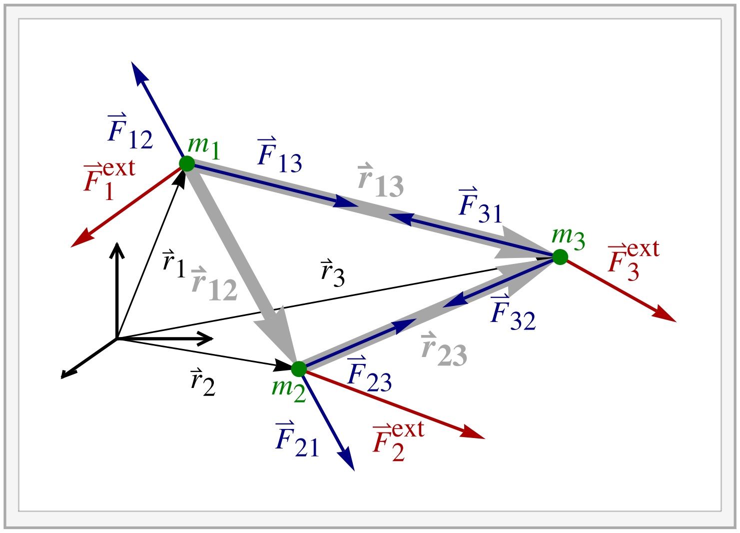
The net torque on a particle (a point) is the torque due to the net force. If the particle is at position $\vec{r}$ then the net torque is $\vec{\tau}_{\text{net}}=\vec{r}\times\vec{F}_{\text{net}}$. It is straightforward to verify that the cross product satisfies the usually product rule for differentiation. Using this we can differentiate the angular momentum for a particle. This gives:

$$\frac{d}{dt}\vec{L}=\left(\frac{d}{dt}\vec{r}\right)\times\vec{p}+\vec{r}\times\frac{d}{dt}\vec{p}=\vec{v}\times m\,\vec{v}+\vec{r}\times\vec{F}_{\text{net}}=\vec{0}+\vec{\tau}_{\text{net}}.$$

This gives the analog of the momentum form of the second law $\vec{F}_{\text{net}}=d\vec{p}\big/dt$. This is

$$\vec{\tau}_{\text{net}}=\frac{d}{dt}\vec{L}.$$

A System of Particles



Interactive Figure

In the preceding chapter we considered a three particle system with masses m_1, $m_2$and m_3 at positions $\vec{r}_1, \vec{r}_2$ and $\vec{r}_3$. As before, we write the forces on m_1 as a sum of internal forces $\vec{F}_{12}$ and $\vec{F}_{13}$ and external forces $\vec{F}_1^{\text{ext}}$. The cross products of this with $\vec{r}_1$ gives the net torque. The torques for m_2 and m_3 break up similarly.

$$\vec{\tau}_{\text{net},1} = \vec{r}_1 \times \vec{F}_1^{\text{ext}} + \vec{r}_1 \times \vec{F}_{12} + \vec{r}_1 \times \vec{F}_{13} = \frac{d}{dt}\vec{L}_1$$

$$\vec{\tau}_{\text{net},2} = \vec{r}_2 \times \vec{F}_2^{\text{ext}} + \vec{r}_2 \times \vec{F}_{21} + \vec{r}_2 \times \vec{F}_{23} = \frac{d}{dt}\vec{L}_2$$

$$\vec{\tau}_{\text{net},3} = \vec{r}_3 \times \vec{F}_3^{\text{ext}} + \vec{r}_3 \times \vec{F}_{31} + \vec{r}_3 \times \vec{F}_{32} = \frac{d}{dt}\vec{L}_3$$

To concentrate on the bulk motion of our system we sum over these expressions. In the previous case with forces the internal forces canceled due to Newton's third law. Here we need to make an additional assumption that the forces are central forces; this is that $\vec{F}_{12}$, the force of mass 2 on mass 1, is directed parallel (or antiparallel) to the line between the masses.

$$\vec{F}_{12} \parallel \left(\vec{r}_1 - \vec{r}_2\right) \iff \left(\vec{r}_1 - \vec{r}_2\right) \times \vec{F}_{12} = \vec{0}$$

Now when we sum the net torques we get a cancellation of the internal torques. The internal torques cancel for all pairs of charges. The cancellation between m_1 and $m_2$follows from

$$\vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21} = \vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \left(-\vec{F}_{12}\right) = \left(\vec{r}_1 - \vec{r}_2\right) \times \vec{F}_{12} = \vec{0}.$$

The other internal forces cancel similarly. We end up with

$$\vec{\tau}_1^{\text{ext}} + \vec{\tau}_2^{\text{ext}} + \vec{\tau}_3^{\text{ext}} = \frac{d}{dt}\left(\vec{L}_1 + \vec{L}_2 + \vec{L}_3\right).$$

It should be clear how this could be generalized to four, or an arbitrary number, of particles. This gives the very fundamental result that for a system of particles

$$\vec{\tau}_{\text{net}}^{\text{ext}} = \frac{d}{dt}\vec{L}_{\text{tot}}.$$

Conservation of Angular Momentum

The conservation of angular momentum follows from the expression above. If there are no external torques on a system then the total angular momentum of the system is conserved.

$$\vec{\tau}_{\text{net}}^{\text{ext}} = \vec{0} \implies \frac{d}{dt}\vec{L}_{\text{tot}} = \vec{0} \implies \Delta \vec{L}_{\text{tot}} = \vec{0}$$

This derivation mirrors the conservation of linear momentum.

This is a very fundamental result. It has deep implications on the very large scale; in astrophysics it is crucial in the dynamics of planets, stars, solar systems and galaxies. It is also important on the very small scale; in particle accelerators where elementary particles are collided and created, angular momentum is always conserved.

More on Rigid Bodies

Axes and Origins

We began with a discussion of rigid bodies rotating about a fixed axis. Then we considered quantities like angular velocity, angular acceleration, torque and angular momentum as vectors. How are the two points of view related? Torque and angular momentum vectors are relative to an origin, where the position vector $\vec{r}$ is based at the origin. If the origin is chosen as some point on the axis then the vector relative to the axis is just the component in the direction of the axis. The torque about some origin is the vector

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

The torque about the z axis is just the z component of this $\tau_z = \tau$ where

$$\tau = r\, F_\perp = r\, F \sin\theta = r_\perp\, F.$$

Similarly, the angular momentum of a particle relative to an origin

$$\vec{L} = \vec{r} \times \vec{p}$$

can be written relative to an axis. If the axis is the z axis then L about the axis is just the z component of L about the origin. $L_z = L$ where

$$L = r\, p_\perp = r\, p \sin\theta = r_\perp\, p.$$

Angular Momentum of a Rigid Body

As before, we view our rigid body as a collection of point masses where the perpendicular distance form the axis to m_i is r_i. Since all the r_i are fixed we get the momentum related to the tangential velocity, which is then related to the angular velocity.

$$p_{i\perp} = m_i\, v_{it} = m_i\, r_i\, \omega$$

The angular momentum of the i^{th} mass becomes

$$L_i = r_i\, p_{i\perp} = r_i\, m_i\, v_{it} = m_i\, r_i^2\, \omega$$

The total angular momentum is the sum over all these terms $L = \sum_i L_i$. Using $I = \sum_i m_i\, r_i^2$ we get the angular momentum of a rotating rigid body

$$L = I\, \omega.$$

This is the result we had in our table relating rotations about a fixed axis to one dimensional linear motion.

The Second Law

We can now, finally, derive the rotational equivalent of the second law $\tau_{\text{net}} = I\, \alpha$. Start with the momentum form of the second law.

$$\vec{\tau}_{\text{net}}^{\text{ext}} = \frac{d}{dt} \vec{L}_{\text{tot}}$$

Now take the component along the axis of rotation. When the system is the rigid body then the net external torque on the rigid body is just the torque on it. Similarly, the total angular momentum is just $I\, \omega$ the angular momentum of the body. We get

$$\tau_{\text{net}} = \frac{d}{dt} L.$$

Using $L = I\,\omega$ and $\alpha = d\omega/dt$ we get our result.

$$\tau_{\text{net}} = I\,\alpha$$

The Torque Due to Gravity

We saw earlier that to calculate the potential energy due to gravity we treat the object as if all the mass is at the center of mass. The same is true for finding the torque due to gravity.

$$\vec{\tau}_{\text{grav}} = \vec{r}_{\text{cm}} \times M\,\vec{g}$$

It is straightforward to verify this. Write the torque as the sum over the torques on all the masses in the body. Then use the definition of center of mass to get the result.

$$\vec{\tau}_{\text{grav}} = \sum_i \vec{r}_i \times m_i\,\vec{g} = \left(\sum_i m_i\,\vec{r}_i\right) \times \vec{g} = \left(M\,\vec{r}_{\text{cm}}\right) \times \vec{g} = \vec{r}_{\text{cm}} \times M\,\vec{g}$$

Equilibrium

The Conditions for Equilibrium

If a body is in equilibrium then there is no acceleration and there is no angular acceleration. This implies that the net force and the net torque *must* vanish.

$$\vec{F}_{\text{net}} = \vec{0} \quad \text{and} \quad \vec{\tau}_{\text{net}} = \vec{0}$$

For the examples we will consider all possible rotation will be in a plane and thus we only need to consider torques relative to an axis.

The Origin (or Axis) is Arbitrary

When considering an equilibrium problem sometimes the choice of axis is clear. Often in isn't clear, though, and there isn't a natural choice. The key point is that the choice of origin or axis is arbitrary. When something is arbitrary then we have the luxury of making a choice that simplifies the problem.

The basic result is this: If the torques balance about one origin and the forces balance then the torques balance about any origin. If the vector from one origin to another is $\vec{r}_0$. If $\vec{r}_i'$ is the vector from the new origin at $\vec{r}_0$ to the mass m_i and $\vec{r}_i$ is from the first origin to the mass then

$$\vec{r}_i = \vec{r}_i' + \vec{r}_0.$$

Take the net torques about these axes to be $\vec{\tau}_{\text{net}}$ and $\vec{\tau}_{\text{net}}'$. If $\vec{\tau}_{\text{net}} = \vec{0}$ and $\vec{F}_{\text{net}} = \vec{0}$ then $\vec{\tau}_{\text{net}}' = \vec{0}$.

$$\vec{0} = \vec{\tau}_{\text{net}} = \sum_i \vec{r}_i \times \vec{F}_i = \sum_i \vec{r}_i' \times \vec{F}_i + \vec{r}_0 \times \sum_i \vec{F}_i = \vec{\tau}_{\text{net}}' + \vec{0}$$

This proves our result.