Chapter G

Momentum and Systems of Particles

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Momentum and Newton's Second Law

So far we have written the second law in terms of the acceleration of a particle. It turns out that Newton wrote it differently; his preferred form of the second law was written in terms of the time derivative of the momentum of the body.

Definition of Momentum

Momentum is a vector quantity describing the dynamics of a moving body. It is defined simply as the mass times the velocity and we use the symbol $\vec{p}$ to denote it.

$$\vec{p} = m\,\vec{v}$$

We will often refer to this as *linear momentum*; this will distinguish it from angular momentum which will be discussed later.

The Second Law

If we take the time derivative of the momentum then, using the product rule, we get

$$\frac{d}{dt}\vec{p} = \left(\frac{d}{dt}m\right)\vec{v} + m\left(\frac{d}{dt}\vec{v}\right) = m\,\vec{a} + \left(\frac{d}{dt}m\right)\vec{v}.$$

Here we are considering a more general possibility, that the mass of a body may change with time. In the case of a constant mass we get

$$\frac{d}{dt}m = 0 \implies \frac{d}{dt}\vec{p} = m\,\vec{a}.$$

This allows us to rewrite the second law in terms of the time derivative of the momentum.

$$\vec{F}_{\text{net}} = \frac{d}{dt}\vec{p}$$

This form is equivalent to the $m\,\vec{a}$ form when the mass is constant. When the mass is changing which form should we use? This new momentum form is the proper one.

Impulse and Momentum

Impulse

Collisions are usually quick things but they are not instantaneous. When a bat hits a baseball, the ball rides along the bat for a period of time. An impulsive force is a large force acting over a short period of time. We will define the impulse as the integral of the force over time. If we take the collision to be between t_i and t_f the force is zero other than in that time interval.

Define the impulse as the integral of the force over the time of some collision.

$$\vec{I} = \int_{t_i}^{t_f} \vec{F}\, dt$$

For a one dimensional force consider a graph of F vs. t. Since the definite integral is the area under a curve the impulse has the interpretation of the area under the force vs. time graph. For two or three dimensions the x-component of the impulse is the integral of the x-component of the force

Average Force

Generally, the average of a function over an interval is the integral of the function over the interval divided by the width of the interval. For the force the integral is the impulse and the width of the interval is $\Delta t = t_f - t_i$. The average force is

$$\overline{\vec{F}} = \frac{\int_{t_i}^{t_f} \vec{F}\, dt}{\Delta t} \quad \text{or} \quad \vec{I} = \overline{\vec{F}}\, \Delta t$$

The Impulse-Momentum Theorem

The impulse-momentum theorem is an immediate consequence of Newton's second law and the fundamental theorem of calculus.

$$\vec{I}_{\text{net}} = \Delta \vec{p}$$

Here the net impulse I_{net} is the impulse of the net force $\vec{F}_{\text{net}}$. The physical significance of this is when there is an impulsive force, it typically is much larger than any other forces acting over the short time of the collision. For example, when a bat hits a baseball that force is much larger than gravity or some other force during the collision.

$$\vec{I}_{\text{net}} = \int_{t_i}^{t_f} \vec{F}_{\text{net}}\, dt = \int_{t_i}^{t_f} \frac{d}{dt}\vec{p}\ dt = \vec{p}(t_f) - \vec{p}(t_i) = \Delta \vec{p}$$

In the above proof, the first equality is the definition of impulse due to the net force. The second uses the second law and the third is the fundamental theorem of calculus.

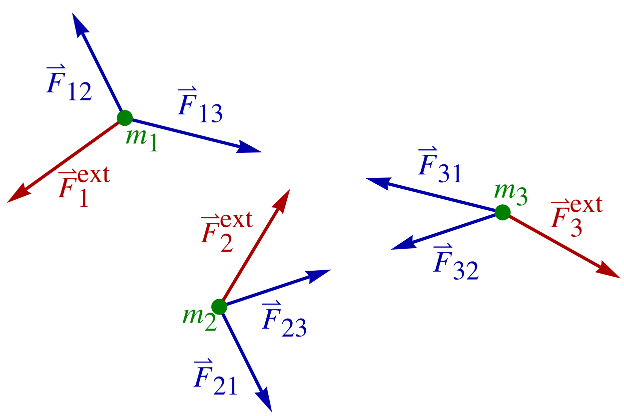
$$\Delta \vec{p} = m\left(\vec{v}_f - \vec{v}_i\right)$$

A System of Particles

So far our discussion of dynamics has applied only to point particles. When we discussed the dynamics of extended objects we treated them as particles. Why was this proper? In all the examples considered the body was not rotating, so each point on it had the same acceleration. This allowed us to treat it as a particle. If a body rotates then different points have different accelerations and we must be more careful. We can treat it as a system of particles.

A system is a collection of point particles. This could represent a huge number of particles, like every atom in a solid or fluid or it could be a small number like the earth, moon and sun. The point is we arbitrarily divide the world into a system and everything else. We then break up the force into internal forces, which are between particles of our system and external force between particles of our system and outside.

A Three Particle System



Consider a three particle system with masses m_1, $m_2$and m_3. For the forces on $m_1$can be written as a sum of internal forces $\vec{F}_{12}$ and $\vec{F}_{13}$ and external forces $\vec{F}_1^{\text{ext}}$, representing everything outside our system acting on m_1. The forces for m_2 and m_3 break up similarly giving

$$\vec{F}_{\text{net},1} = \vec{F}_1^{\text{ext}} + \vec{F}_{12} + \vec{F}_{13} = \frac{d}{dt}\vec{p}_1 = m_1\,\vec{a}_1$$

$$\vec{F}_{\text{net},2} = \vec{F}_2^{\text{ext}} + \vec{F}_{21} + \vec{F}_{23} = \frac{d}{dt}\vec{p}_2 = m_2\,\vec{a}_2$$

$$\vec{F}_{\text{net},3} = \vec{F}_3^{\text{ext}} + \vec{F}_{31} + \vec{F}_{32} = \frac{d}{dt}\vec{p}_3 = m_3\,\vec{a}_3.$$

To concentrate on the bulk motion of our system we sum over these expressions. The crucial point is that the internal forces cancel by Newton's third law. $\vec{F}_{12} + \vec{F}_{21} = \vec{0}$, $\vec{F}_{13} + \vec{F}_{31} = \vec{0}$ and $\vec{F}_{23} + \vec{F}_{32} = \vec{0}$.

$$\vec{F}_1^{\text{ext}} + \vec{F}_2^{\text{ext}} + \vec{F}_3^{\text{ext}} = \frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2 + \vec{p}_3\right) = m_1\,\vec{a}_1 + m_2\,\vec{a}_2 + m_3\,\vec{a}_3$$

The General System

For a general system we define $\vec{F}_{\text{net}}^{\text{ext}}$ as the net force

$$\vec{F}_{\text{net}}^{\text{ext}} = \vec{F}_1^{\text{ext}} + \vec{F}_2^{\text{ext}} + \ldots = \sum_i \vec{F}_i^{\text{ext}},$$

$\vec{p}_{\text{tot}}$ as the total momentum and M as the total mass

$$\vec{p}_{\text{tot}} = \vec{p}_1 + \vec{p}_2 + \ldots = \sum_i \vec{p}_i$$

$$M = m_1 + m_2 + \ldots = \sum_i m_i.$$

The center of mass will be defined in the next section so that

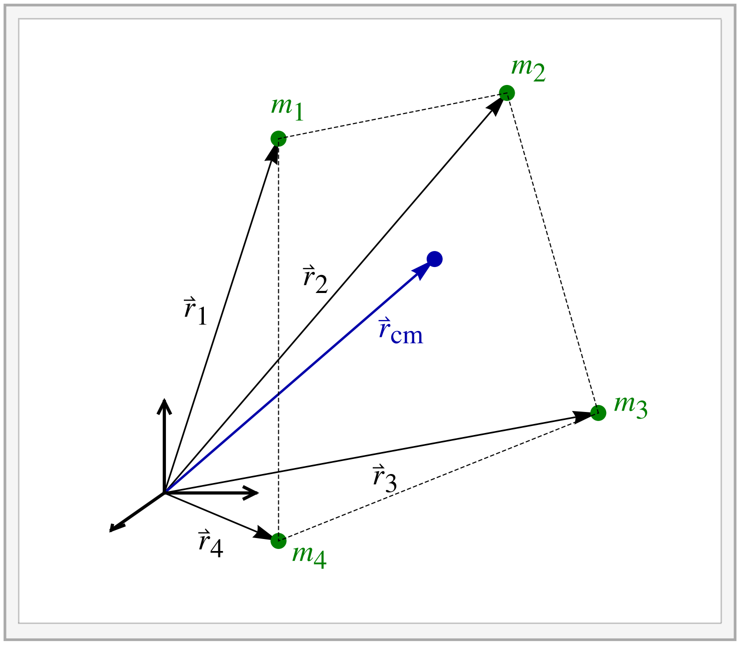
$$M\,\vec{a}_{\text{cm}} = m_1\,\vec{a}_1 + m_2\,\vec{a}_2 + \ldots = \sum_i m_i\,\vec{a}_i.$$

With this we get the two expressions for the second law for a system of particles.

$$\vec{F}_{\text{net}}^{\text{ext}} = \frac{d}{dt}\vec{p}_{\text{tot}}$$

$$\vec{F}_{\text{net}}^{\text{ext}} = M\,\vec{a}_{\text{cm}}$$

Center of Mass



Interactive Figure

We define the center of mass so that $M\,\vec{a}_{\text{cm}} = m_1\,\vec{a}_1 + m_2\,\vec{a}_2 + \ldots = \sum_i m_i\,\vec{a}_i$. Since the second derivative of the position vector $\vec{r}$ is the acceleration $\vec{a}$ we define the center of mass by

$$\vec{r}_{\text{cm}} = \frac{1}{M}\left(m_1\,\vec{r}_1 + m_2\,\vec{r}_2 + \ldots\right) = \frac{1}{M}\sum_i m_i\,\vec{r}_i.$$

The vector $\vec{r}_i$ is the position vector of the mass m_i; it is the vector pointing from the origin of the coordinate system to the mass. The x component of the position vector is just x. The x component of the center of mass is thus

$$x_{\text{cm}} = \frac{1}{M}(m_1\,x_1 + m_2\,x_2 + \ldots) = \frac{1}{M}\sum_i m_i\,x_i.$$

The y and z components satisfy similar expressions.

Conservation of Linear Momentum

The conservation of momentum is a consequence of the momentum form of the second law for a system.

$$\vec{F}^{\text{ext}}_{\text{net}} = \frac{d}{dt}\vec{p}_{\text{tot}}$$

If there are no external forces acting on a system them the total momentum of the system is conserved.

$$\vec{F}^{\text{ext}}_{\text{net}} = \vec{0} \implies \frac{d}{dt}\vec{p}_{\text{tot}} = \vec{0} \implies \Delta\,\vec{p}_{\text{tot}} = \vec{0}$$

This is the second fundamentally conserved quantities encountered in our course. To see how this is fundamental, imagine enlarging the system to include everything; there is then, by definition, no external force and thus the total momentum is conserved.

The conservation is also true component by component. If there is no external force in some direction, say the x direction, then the x component of the total momentum is conserved.

$$F^{\text{ext}}_{\text{net},x} = 0 \implies \frac{d}{dt}p_{\text{tot},x} = 0 \implies \Delta\,p_{\text{tot},x} = 0$$

Symmetries and Conservation Laws - Noether's Theorem

It is a very deep and fundamental matter that symmetries give rise to conserved quantities. This result is known as Noether's theorem. The mathematician Amalie Noether demonstrated around 1920 that to every symmetry there is a conservation law. For example, the invariance or symmetry of the laws of physics under time translations, that the laws are the same now as a few minutes from now, implies that there is a conserved quantity; this is energy! The symmetry that the laws of physics are invariant under spatial translations implies a conserved quantity, in this case linear momentum. Rotational symmetry implies conservation of angular momentum.

The Center of Mass Frame and Energy

Taking the derivative of the definition of the center of mass gives

$$\vec{p}_{\text{tot}} = M\,\vec{v}_{\text{cm}}.$$

Recall the discussion of relative motion: $\vec{v}$ is the velocity of something with respect to a fixed frame, $\vec{v}'$ is the velocity with respect to a moving frame and $\vec{v}_0$ is the velocity of the moving frame. These are related by $\vec{v} = \vec{v}' + \vec{v}_0$. The center of mass frame is the frame that moves with the center of mass.

$$\vec{v}_0 = \vec{v}_{\text{cm}} \implies \vec{v} = \vec{v}' + \vec{v}_{\text{cm}}$$

In the center of mass frame it follows that $\vec{p}'_{\text{tot}} = \vec{0}$ and $\vec{v}'_{\text{cm}} = \vec{0}$. In other words, in the center of mass frame the center of mass is at rest.

The total kinetic energy of any system is

$$K_{\text{tot}} = \sum_i \frac{1}{2} m_i v_i^2$$

Let us now consider kinetic energy with respect to the center of mass frame. Here we use $\vec{v}_i = \vec{v}_i' + \vec{v}_{\text{cm}}$ to write $v_i^2 = v_i'^2 + v_{\text{cm}}^2 + 2\,\vec{v}_{\text{cm}} \cdot \vec{v}_i'$. Inserting this into the general expression for total kinetic energy gives

$$K_{\text{tot}} = \sum_i \frac{1}{2} m_i v_i'^2 + \sum_i \frac{1}{2} m_i v_{\text{cm}}^2 + \sum_i m_i\ \vec{v}_{\text{cm}} \cdot \vec{v}_i' .$$

We can factor the constant terms out of the sums giving

$$K_{\text{tot}} = \sum_i \frac{1}{2} m_i v_i'^2 + \frac{1}{2} \left(\sum_i m_i \right) v_{\text{cm}}^2 + \vec{v}_{\text{cm}} \cdot \sum_i m_i \vec{v}_i' .$$

Using $M = \sum_i m_i$ and $\vec{p}_{\text{tot}}' = \sum_i m_i \vec{v}_i' = \vec{0}$ we end up with a simple result.

$$K_{\text{tot}} = K_{\text{tot,cm}}' + \frac{1}{2} M v_{\text{cm}}^2$$

$$\text{where } K_{\text{tot,cm}}' = \sum_i \frac{1}{2} m_i v_i'^2 \text{ with } \vec{v}_i' = \vec{v}_i - \vec{v}_{\text{cm}}$$

This expression is simple to interpret. $K_{\text{tot,cm}}'$ is the kinetic energy in the center of mass frame. $(1/2) M v_{\text{cm}}^2$ is the energy of the bulk motion of the center of mass.

Two-Body Collisions

We now consider two-body collisions as a special case of our more general discussion. If there are no external forces act while two bodies collide, then the total momentum of the two-body system is conserved. Even if there are external forces, often we can neglect them and consider momentum conserved. Consider a mid-air collision between two bodies. Gravity acts as an external force during the collision but usually, to a good approximation, the collision is so fast that the large impulsive internal forces dominate the gravity force and gravity can be neglected. We can equate the total momentum just before and just after the collision.

Momentum Conservation

Mass m_1 moving at $\vec{v}_{1i}$ collides with mass m_2 moving at $\vec{v}_{2i}$. After the collision the velocities are $\vec{v}_{1f}$ and $\vec{v}_{2f}$. The conservation of momentum for a two-body collision has the form

$$m_1\,\vec{v}_{1i} + m_2\,\vec{v}_{2i} = m_1\,\vec{v}_{1f} + m_2\,\vec{v}_{2f}.$$

The left-hand side is the total initial momentum and the right hand side is the total final momentum.

In the case of a one dimensional collision then the above expression applies but we may omit the vector arrows. In one dimension a vector is a real number and the sign gives the direction. The vector nature of momentum and velocity is reflected in their signs.

Elastic Collisions - Kinetic Energy Conservation

Typically energy is lost in a collision. Often to a reasonable approximation we can consider conservation of energy. The relevant energy is kinetic.

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad \text{(elastic)}$$

In other words

$$K_{\text{tot},i} = K_{\text{tot},f}. \quad \text{(elastic)}$$

Inelastic and Totally Inelastic Collisions

When we say a collision is inelastic we mean merely that it is not elastic.

$$K_{\text{tot},i} \neq K_{\text{tot},f} \quad \text{(inelastic)}$$

The extreme case of an inelastic collision is called totally inelastic. The most energy that can be lost in a collision is when the center of mass energy vanishes $K'_{\text{tot,cm}} = 0$. This means there is no relative motion in the center of mass frame. With two bodies this means that they have the same final velocity.

$$K'_{\text{tot,cm}} = 0 \iff \vec{v}_{1f} = \vec{v}_{2f} = \vec{v}_f \quad \text{(totally inelastic)}$$

The conservation of momentum formula then has the simple form:

$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = (m_1 + m_2)\, \vec{v}_f. \quad \text{(totally inelastic)}$$

One Dimensional Elastic Collisions

For the case of a one dimensional elastic collision we can solve for the final velocities in terms of the initial velocities and the masses.

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \quad \text{(momentum eq.)}$$

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad \text{(energy eq.)}$$

Consider a graph of v_{2f} vs. v_{1f}. With the masses and initial velocities given the momentum equation becomes a line and the kinetic energy equation becomes an ellipse. It follows that there are two solutions for (v_{1f}, v_{2f}). To find this one could use the momentum equation to solve for one variable in terms of the other and insert that into the kinetic energy equation. This gives a quadratic expression for the remaining unknown. With this we can, eventually, get the two solutions. Of the two solutions one is a trivial solution and will always be ignored. It is clear that

$$v_{1f} = v_{1i} \text{ and } v_{2f} = v_{2i}$$

is a trivial solution to our problem; it corresponds to the two masses never colliding and will be ignored. We are left with the one physically important solution. This solution method awkward and tedious. We will now derive a simplified method for solving such problems.

Take the kinetic energy equation, multiply by two and put the m_1 terms on the left and the m_2 terms on the right.

$$m_1 \left(v_{1i}^2 - v_{1f}^2\right) = m_2 \left(v_{2f}^2 - v_{2i}^2\right)$$

Similarly, put the momentum equation in the form with m_1 terms on the left and the m_2 terms on the right.

$$m_1 \left(v_{1i} - v_{1f}\right) = m_2 \left(v_{2f} - v_{2i}\right)$$

Now divide the second expression into the first. Using $a^2 - b^2 = (a - b)(a + b)$ we get

$$v_{1i} + v_{1f} = v_{2i} + v_{2f}.$$

We then replace the quadratic kinetic energy expression with the above linear expression. To solve the problem we then use this equation with the momentum equation.

Rocket Propulsion

When moving with respect to a medium one can propel something forward by pushing backward. To walk, you push backward in the floor and the floor then pushes forward on you. A boat pushes backward on the water and the water pushes forward in it. A plane or jet propels itself similarly by pushing backward on the air. This leads to an obvious question for rocket propulsion. How does a rocket propel itself forward in the vacuum of space? The answer is that the rocket throws part of itself, the spent fuel, backward and thus propels the rest of the rocket forward. The mass change of a rocket is essential to its propulsion.

Consider a rocket of mass M moving with a velocity v. The rocket propels itself forward by shooting spent fuel backward at a speed of v_e, the exhaust speed. In doing this the mass of the rocket changes. If dM is the infinitesimal change in the rocket's mass, then since $dM < 0$ the (positive) mass of ejected fuel is $|dM|$. Ejecting the fuel backward makes the rocket recoil forward by an infinitesimal dv. Looking at conservation of momentum in the frame where the rocket was initially at rest gives

$$0 = M\,dv - v_e\,|dM| = M\,dv + v_e\,dM \implies dv = -v_e\,\frac{dM}{M}.$$

We can integrate this expression and get

$$\int_{v_i}^{v_f} dv = -v_e \int_{M_i}^{M_f} \frac{dM}{M} \implies v_f - v_i = -v_e\left(\ln M_f - \ln M_i\right).$$

This leads us to our result relating the mass change of a rocket and the exhaust speed to the gain in velocity of the rocket.

$$v_f - v_i = v_e \ln \frac{M_i}{M_f}$$