Chapter C

Vectors and Two Dimensional Kinematics

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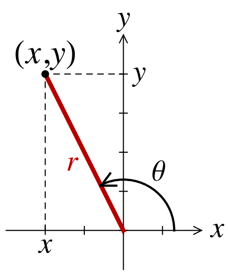
Vector Algebra - I

Polar Coordinates

(x, y) are the Cartesian (or rectangular) coordinates of some point on a plane. r and θ are the polar coordinates; $r = \sqrt{x^2 + y^2}$ is the distance from the origin to the point and θ is the angle measured counterclockwise from the positive x axis to the point. The definitions of the trig functions, for general angles, are given by

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r} \quad \text{and} \quad \tan\theta = \frac{y}{x}.$$

Using the above definitions it is a straightforward matter to find the formulas for converting between polar and rectangular coordinates.



r* and *θ* ⟹ *x* and *y

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

x* and *y* ⟹ *r* and *θ

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & \text{when } x > 0 \\ 180° + \tan^{-1}\left(\frac{y}{x}\right) & \text{when } x < 0 \end{cases}$$

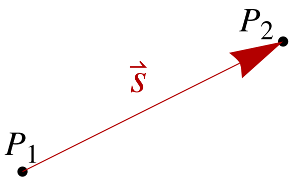
The subtlety in solving for the angle follows from the identity: $\tan\theta = \tan(180° + \theta)$. The range of the $\tan^{-1}$ (also written arctan) function is $-90° < \theta < 90°$, which corresponds to the quadrants I and IV; this is equivalent to the condition that $x > 0$. The case of quadrants II and III, when $x < 0$, requires shifting the result of the inverse tangent by $180°$.

Vector Basics

A vector is a quantity with both a magnitude and a direction. A scalar has only a magnitude; it is just a real number. The magnitude of a vector is a nonnegative (positive or zero) scalar. Velocity is a vector quantity and speed is its magnitude. Acceleration, force and momentum are also vectors. Time, temperature, mass and pressure are examples of scalars.

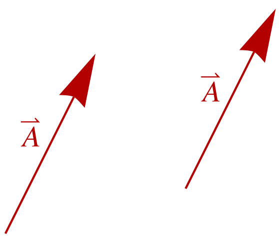
We will write a vector variable by a symbol with an "⇀" over it. The magnitude of a vector is given by the symbol without the arrow or by applying the "|| ||" brackets to the vector. If $\vec{A}$ is some vector then $A = ||\vec{A}||$ is its magnitude.

We can represent vectors by arrows. Suppose someone walks from a starting point P_1 to a stopping point P_2. A displacement vector $\vec{s}$ (or $\Delta\vec{r}$) may be viewed as an arrow with its tail at P_1 and its tip at P_2.



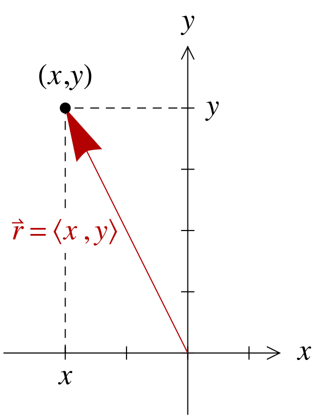
A position vector $\vec{r}$ is a displacement vector with its tail at the origin. For more general vectors we represent them by arrows pointing in the direction of the vectors and the length of the arrow is proportional to the magnitude of the vector. For instance, if one velocity vector has a magnitude of 60 mi/hr and another has 30 mi/hr then the arrow representing the first should have twice the length of the second.

A vector has no fixed position. If a vector arrow is moved keeping its length and direction fixed then it still is the same vector.



Component Definition - Position and Displacement Vectors

A position vector is a way to label a position in the Cartesian plane; it has its tail at the origin and its head at the position it labels. We will use an angled bracket notation for vectors. The position vector that labels the point (x, y) will be written as $\vec{r} = \langle x, y \rangle$.

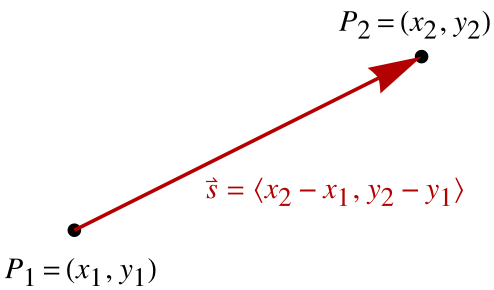


We will define the magnitude and direction of a vector so that the polar coordinates, r and θ, are the magnitude and direction of the vector.

In the Cartesian plane we will denote vector from (x_1, y_1) to (x_2, y_2) by:

$$\vec{s} = \langle x_2 - x_1, y_2 - y_1 \rangle.$$

This is called a displacement vector. A position vector is clearly a displacement vector with its tail at the origin.

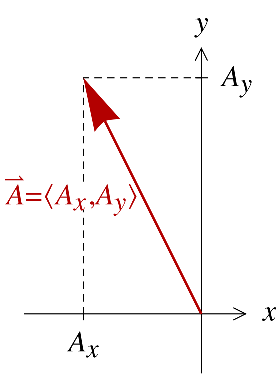


Component Definition - General Vectors

We will write a 2-dimensional vector $\vec{A}$ as a pair of real numbers A_x and A_y called components. (A 3D vector is a triple.) We will use the "angled-bracket" notation for vectors.

$$\vec{A} = \langle A_x, A_y \rangle$$

The component A_x has the interpretation as the amount the vector $\vec{A}$ is in the x-direction and A_y the y-direction.



Magnitude and Direction Angle

We define A_x and A_y as the components of the vector $\vec{A}$. A_x is the part of $\vec{A}$ in the x direction and similarly A_y is the y part. The Cartesian coordinates x and y are the components of a position vector $\vec{r}$. For a two dimensional vector we can represent the direction with an angle, measured as in the polar coordinates. To convert between the magnitude and direction angle and the components of a two dimensional vector we have analogous expressions to the ones for polar coordinates.

A and $\theta \implies A_x$ and A_y

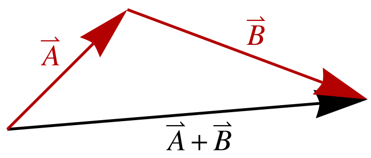
$$A_x = A\cos\theta \text{ and } A_y = A\sin\theta$$

A_x and $A_y \implies A$ and θ

$$A = \sqrt{A_x{}^2 + A_y{}^2} \text{ and } \theta = \begin{cases} \tan^{-1}\left(\frac{A_y}{A_x}\right) \text{ when } A_x > 0 \\ 180° + \tan^{-1}\left(\frac{A_y}{A_x}\right) \text{ when } A_x < 0 \end{cases}$$

Vector Addition

Suppose a displacement vector $\vec{s}_1$ corresponds to someone walking from P_1 to P_2. Suppose that then the person walks from P_2 to P_3; call this displacement $\vec{s}_2$. The net displacement is the vector from P_1 to P_3; this is what we will define as the sum of the two displacements $\vec{s}_1 + \vec{s}_2$. To generalize this to any vectors, we will define the sum of general vectors $\vec{A}$ and $\vec{B}$. Draw the vectors as shown, with the tail of $\vec{B}$ at the tip of $\vec{A}$. The sum the vectors $\vec{A} + \vec{B}$ is the vector drawn from the tail of $\vec{A}$ to the tip of $\vec{B}$.



With our component definition vector addition takes the very simple form:

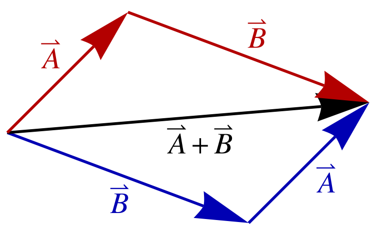
$$\vec{A} + \vec{B} = \langle A_x, A_y \rangle + \langle B_x, B_y \rangle = \langle A_x + B_x, A_y + B_y \rangle.$$

Commutative Property

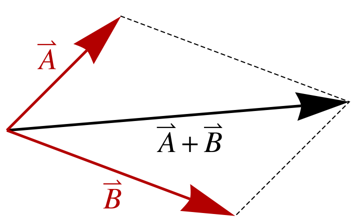
An algebraic operation is commutative when changing the order of the items doesn't affect the result. For vector addition this takes the form.

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

The cummutative property of the addition of reals implies this for vectors.



Because of the commutative property there are two more ways of adding vectors we can consider. In addition to placing the tail of $\vec{B}$ at the tip of $\vec{A}$, we can place the tail of $\vec{A}$ at the tip of $\vec{B}$. Also there is the parallelogram rule: Draw the two vector together tail to tail and complete the parallelogram; the sum it the vector from the common tail of the vectors to the opposite corner.

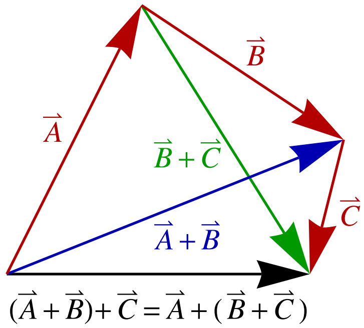


Associative Property

From the definition of vector addition it is clear it satisfies.

$$\left(\vec{A} + \vec{B}\right) + \vec{C} = \vec{A} + \left(\vec{B} + \vec{C}\right)$$

This property is called associativity. For an associative operation it is not necessary to use brackets, since the order of the operation is unimportant.



Identity

The identity vector $\vec{0}$ is the vector that leaves any other vector $\vec{A}$ unchanged under addition.

$$\vec{A}+\vec{0}=\vec{A}$$

It is clear that the zero vector has zeros as components.

$$\vec{0}=\langle 0, 0\rangle$$

The magnitude of the identity vector is 0, $0=\|\vec{0}\|$. Note that the direction of $\vec{0}$ is undefined; in fact, it is the only vector with an undefined direction.

Additive Inverse.

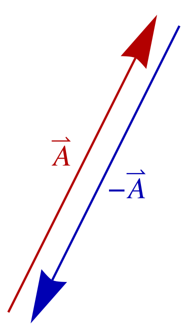
For any vector $\vec{A}$ there is an additive inverse vector $-\vec{A}$ with the property:

$$\vec{A}+\left(-\vec{A}\right)=\vec{0}$$

Clearly, this has the value

$$-\vec{A}=\left\langle -A_x, -A_y\right\rangle$$

and has the same magnitude and is in the opposite direction.



Vector Subtraction

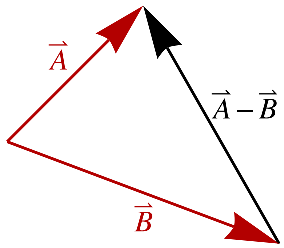
We define vector subtraction by adding the inverse.

$$\vec{A}-\vec{B}=\vec{A}+\left(-\vec{B}\right)$$

In terms of components we have

$$\vec{A} - \vec{B} = \langle A_x, A_y \rangle - \langle B_x, B_y \rangle = \langle A_x - B_x, A_y - B_y \rangle.$$

The simplest way to view the vector $\vec{A} - \vec{B}$ is as the vector that when added to $\vec{B}$ gives $\vec{A}$; if the vectors are drawn tail to tail then it is the vector from the tip of $\vec{B}$ to the tip of $\vec{A}$



Scalar Multiplication

If $\vec{A}$ is a vector and c is a scalar then we can define their product $c\,\vec{A}$ as a vector.

$$c\,\vec{A} = \langle c\,A_x, c\,A_y \rangle$$

It is clear that its magnitude is given by

$$\left\| c\,\vec{A} \right\| = |c|\,||\vec{A}||,$$

where |c| is the absolute value of the scalar. The direction of $c\,\vec{A}$ is the same as $\vec{A}$ when $c > 0$ and opposite to $\vec{A}$ when $c < 0$. When $c = 0$ we get $0\,\vec{A} = \vec{0}$. Note also that $1\,\vec{A} = \vec{A}$ and $(-1)\,\vec{A} = -\vec{A}$.

The scalar multiplication operation has the associative and distributive properties.

Associative Property

$$(c\,d)\,\vec{A} = c\left(d\,\vec{A}\right)$$

Distributive Properties

$$(c + d)\,\vec{A} = c\,\vec{A} + d\,\vec{A} \text{ and } c\left(\vec{A} + \vec{B}\right) = c\,\vec{A} + c\,\vec{B}$$

Unit Vectors and Notation

A unit vector is a vector of magnitude one. We denote unit vectors with a " ^ " over its top. For any vector $\vec{A}$ we can simply find the unit vector in its direction $\hat{A}$ by

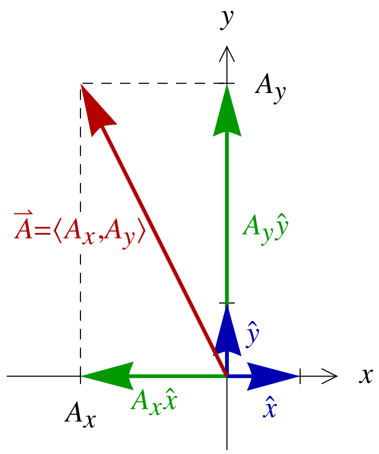
$$\hat{A} = \frac{\vec{A}}{||\vec{A}||}.$$

Basis unit vectors are unit vectors along the coordinate axes. We will use $\hat{x}$ and $\hat{y}$ for the unit vectors in the x and y directions. The more traditional notation for these unit vectors is to write them as $\hat{i}$ and $\hat{j}$.

$$\hat{x} = \hat{i} = \langle 1, 0 \rangle \text{ and } \hat{y} = \hat{j} = \langle 0, 1 \rangle$$

Any vector can then be written in terms of these basis unit vectors

$$\vec{A} = \langle A_x, A_y \rangle = A_x\,\hat{x} + A_y\,\hat{y}$$



3D Vectors

It is straightforward to generalize our two dimensional discussion to a three dimensional one. We need to add $\hat{z} = \hat{k}$ for the unit vector in the z direction.

$$\hat{x} = \hat{i} = \langle 1, 0, 0 \rangle\,,\quad \hat{y} = \hat{j} = \langle 0, 1, 0 \rangle \text{ and } \hat{z} = \hat{k} = \langle 0, 0, 1 \rangle$$

We can similarly write any vector in terms or its components and unit vectors

$$\vec{A} = \langle A_x, A_y, A_z \rangle = A_x\,\hat{x} + A_y\,\hat{y} + A_z\,\hat{z}.$$

The magnitude of a vector is

$$A = \|\vec{A}\| = \sqrt{A_x{}^2 + A_y{}^2 + A_z{}^2}\,.$$

The only subtlety with three dimensions is specifying directions as angles. It takes two angles to specify a direction is 3D; these are the θ and ϕ of spherical coordinates. To avoid these issues the best way to represent a direction for a three dimensional vector is to just write a unit vector in the direction of that vector

$$\hat{A} = \frac{\vec{A}}{\|\vec{A}\|}.$$

2D and 3D Kinematics

The General Problem

Position as a Function of Time

There are two equivalent ways of describing position in two dimensions. One is by labelling the coordinates (x, y). The other is by giving the position vector $\vec{r}$ of the position. The two are related; the coordinates are the components of the position vector. To label position as a function of time we can consider x and y as separate functions of time or as $\vec{r}$ as a function of time.

$$x(t) \text{ and } y(t) \iff \vec{r}(t) = \langle x(t), y(t) \rangle$$

The actual path followed by the body is called the trajectory. It is represented by a plot of a path in the xy plane.

Average Velocity

The average velocity, as we saw in the one dimensional case, refers to two times. At t_i the position vector is $\vec{r}_i = \vec{r}(t_i)$ and at t_f it is $\vec{r}_f = \vec{r}(t_f)$. The displacement is the difference of these two positions

$$\Delta \vec{r} = \vec{r}_f - \vec{r}_i = \langle x_f - x_i, y_f - y_i \rangle.$$

The average velocity vector is then defined as

$$\overline{\vec{v}} = \frac{\Delta \vec{r}}{\Delta t} = \left\langle \frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t} \right\rangle = \langle \overline{v}_x, \overline{v}_y \rangle.$$

Instantaneous Velocity

The instantaneous velocity is defined as the limit of the average velocity as Δt approaches zero; it is the time derivative of the position vector.

$$\vec{v} = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \langle v_x, v_y \rangle.$$

The magnitude of the velocity is called the speed.

$$v = \| \vec{v} \| = \sqrt{v_x^2 + v_y^2} = \text{speed}$$

Average Acceleration

As with the one dimensional case acceleration is to velocity as the velocity is to position.

$$\overline{\vec{a}} = \frac{\Delta \vec{v}}{\Delta t} = \left\langle \frac{\Delta v_x}{\Delta t}, \frac{\Delta v_y}{\Delta t} \right\rangle = \langle \overline{a}_x, \overline{a}_y \rangle.$$

Instantaneous Acceleration

The instantaneous acceleration is similarly defined as a limit of the average acceleration or simply as the time derivative of the velocity.

$$\vec{a} = \frac{d\vec{v}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \left\langle \frac{dv_x}{dt}, \frac{dv_y}{dt} \right\rangle = \langle a_x, a_y \rangle.$$

Constant Velocity and Acceleration

The cases of constant velocity and acceleration follows the one dimensional example. Since we are now dealing with vectors the arbitrary constants introduced in antidifferentiation become vector quantities.

$$\text{const}\, \vec{v} \implies \vec{a} = \vec{0} \text{ and } \vec{r}(t) = \vec{r}_0 + \vec{v}\, t$$

$$\text{const}\, \vec{a} \implies \vec{v}(t) = \vec{v}_0 + \vec{a}\, t \text{ and } \vec{r}(t) = \vec{r}_0 + \vec{v}_0\, t + \frac{1}{2} \vec{a}\, t^2$$

Projectile Motion

An important case of motion with constant acceleration is that of projectile motion. Projectile motion is two dimensional motion of body under the influence of only gravity. Insisting on only gravity means that we ignore air resistance. Spin effects like a curving baseball or a slicing golf ball are associated with air resistance and will also be ignored.

For 2D motion with constant acceleration we can modify the four constant acceleration equations into two sets of four equations.

***x* Equations**	***y* Equations**
$v_x = v_{0x} + a_x t$	$v_y = v_{0y} + a_y t$
$\Delta x = \frac{1}{2}(v_x + v_{0x})\, t$	$\Delta y = \frac{1}{2}(v_y + v_{0y})\, t$
$\Delta x = v_{0x} t + \frac{1}{2} a_x t^2$	$\Delta y = v_{0y} t + \frac{1}{2} a_y t^2$
$v_x^2 - v_{0x}^2 = 2 a_x \Delta x$	$v_y^2 - v_{0y}^2 = 2 a_y \Delta y.$

For projectile motion we will take x as the horizontal direction and y as the vertical direction. The acceleration due to gravity is downward and of magnitude g. Writing this as a vector gives

$$\vec{a} = -g\,\hat{y} = \langle 0, -g \rangle \quad \text{or} \quad a_x = 0 \text{ and } a_y = -g$$

Inserting these components into the two sets of equations above gives:

Horizontal Equations	**Vertical Equations**
$v_x = v_{0x}$	$v_y = v_{0y} - g\, t$
$\Delta x = v_{0x} t$	$\Delta y = \frac{1}{2}(v_y + v_{0y})\, t$
	$\Delta y = v_{0y} t - \frac{1}{2} g t^2$
	$v_y^2 - v_{0y}^2 = -2 g \Delta y.$

The horizontal motion is simple. Since there is no horizontal (the x direction) acceleration, the x component of the velocity is constant. The vertical part (the y direction) of the motion is equivalent to free fall. The key to solving projectile motion problems is keeping the two parts separate.

If a projectile is launched at an initial angle of θ with an initial speed v_0 then the components of the initial velocity are given by

$$v_{0x} = v_0 \cos\theta \text{ and } v_{0y} = v_0 \sin\theta.$$

To solve for the trajectory we can choose, for simplicity, that the motion begins at the origin $x_0 = 0 = y_0$ giving $\Delta x = x$ and $\Delta y = y$. Solving the horizontal equation for time gives $t = x / v_{0x}$. Inserting this into the vertical expression $y = v_{0y} t - \frac{1}{2} g t^2$ gives

$$y = \frac{v_{0y}}{v_{0x}} x - \frac{g}{2 v_{0x}^2} x^2.$$

It is clear that this is a parabola.

The *range R* of a projectile is the total horizontal distance travelled in the air when it returns to its original level, $y = 0$ in the expression above. We can then factor out an x from the expression and solve for x.

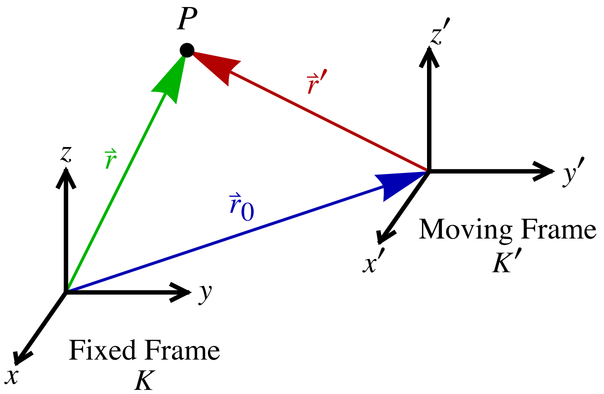
$$x = 0 \text{ and } x = \frac{2 v_{0x} v_{0y}}{g}.$$

The $x = 0$ solution is trivial. Setting the other to R, writing the components in terms of v_0 and θ, and using the identity $\sin 2\theta = 2 \sin\theta \cos\theta$ gives an expression for the range.

$$R = \frac{v_0^2}{g} \sin 2\,\theta$$

Relative Motion

A frame of reference is some coordinate system used to study motion. Suppose there is a fixed frame K and a moving frame K', that moves with a velocity $\vec{v}_0$ with respect to K. If we take a moving body to be at point P. If $\vec{r}$ is the vector from the origin of K to P, $\vec{r}'$ is the vector from the origin of K' to P, and $\vec{r}_0$ is the vector from the origin of K to the origin of K'.



It follows that the three position vectors are related by

$$\vec{r} = \vec{r}' + \vec{r}_0.$$

We want to relate the velocities of the moving body with respect to these two frames. The velocities with respect to K and $K'$are $\vec{v}$ and $\vec{v}'$, respectively. We can relate the velocities in these two frames by taking the time derivative of the expression above:

$$\vec{v} = \vec{v}' + \vec{v}_0,$$

where $\vec{v}_0$ is the velocity of the moving frame.

As an example suppose an object is thrown from a moving car. Take the fixed frame K to be the frame of the road and K' is the frame of the car which moves at the velocity $\vec{v}_0$ with respect to the road. The object moves at $\vec{v}'$ with respect to the car and at $\vec{v}$ with respect to the road.