Chapter B

One Dimensional Kinematics

Blinn College - Physics 2425 - Terry Honan

Kinematics is the study of motion. This chapter will introduce the basic definitions of kinematics. The definitions of the velocity and acceleration will require the introduction of the basic notions of calculus, most specifically the derivative. We will also consider in detail the simple special cases of motion with constant velocity and constant acceleration. Free fall will be discussed as an example of motion with constant acceleration.

The General Problem

By one dimensional motion we mean motion constrained to a line. As examples consider a car driving on a straight road or the vertical motion of an elevator. The problem of motion in two or three dimensions will be discussed in the next chapter.

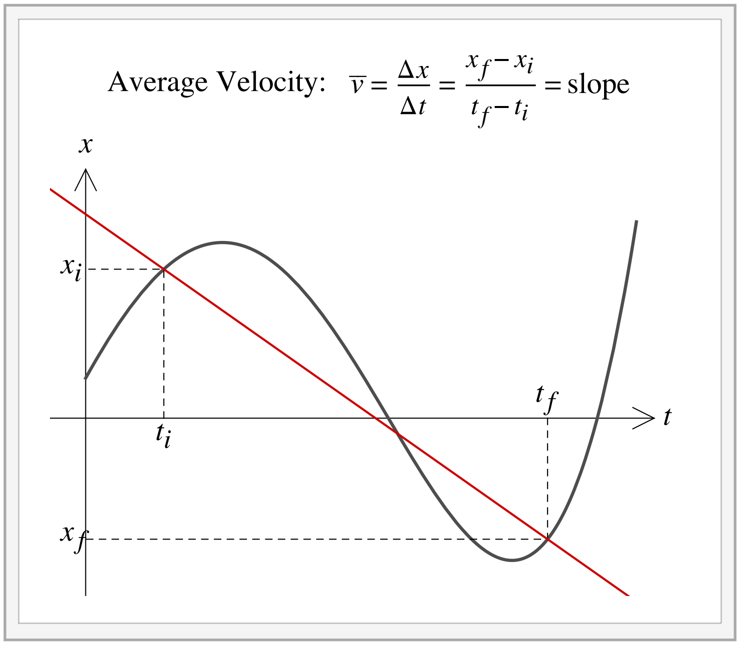
Position as a Function of Time

To mathematically define position we need to attach a real number line (the x-axis) to the line of motion. To do this there are two arbitrary choices; we mush choose the $x = 0$ position and then we must choose the positive direction.

Any 1D motion can be represented graphically. Time is the independent variable, so it will be the horizontal axis. We will then consider graphs of x as a function of time, where x is the vertical axis..

Velocity and the Derivative

Average Velocity



Interactive Figure

If a car drives 130 mi in 2 hours, we can calculate a velocity of 65 mi/hr. This is not necessarily what the speedometer would read; the speedometer reads the magnitude of the instantaneous velocity. In this case 65 mi/hr is what we call the average velocity.

We will define the average velocity by

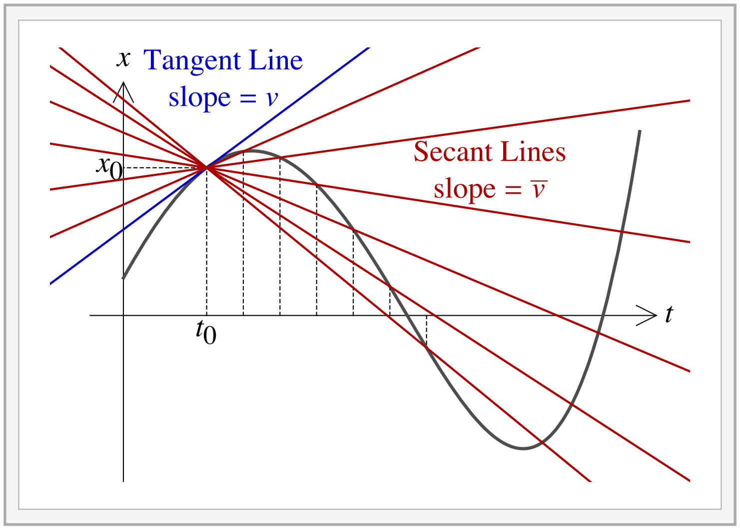
$$\overline{v} = \frac{\Delta x}{\Delta t}$$

where Δ (Delta) generally will represent the final value minus the initial

$$\Delta x = x_f - x_i \text{ and } \Delta t = t_f - t_i.$$

Note that the average velocity corresponds to two times t_i and t_f, and x_i and x_f are the positions at the two times. In a graph of x vs. t the average velocity has the interpretation as the slope of the secant line between the two points (t_i, x_i) and (t_f, x_f).

Instantaneous Velocity



Interactive Figure

The instantaneous velocity refers to a single time t. Take the position at t to be x. We can then consider a later time $t + \Delta t$, where the position is $x + \Delta x$. The average velocity between these two times is $\Delta x / \Delta t$. To get the instantaneous velocity we let Δt become small; we do this by taking the limit as $\Delta t \to 0$. This gives the derivative of calculus; instantaneous velocity is the time derivative of position.

$$v = \frac{dx}{dt} = \dot{x} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

The graphical interpretation of the instantaneous velocity is simple. The average velocity is the slope of the secant lines. If we consider the secant lines corresponding to different Δt values then as $\Delta t \to 0$, these secant lines approach the tangent line. The velocity at t is then the slope of that tangent line. When we refer to the slope of a graph at some time, we mean the slope of a line tangent to the graph at that time.

Acceleration

Acceleration is to velocity as the velocity is to the position. Velocity is the time derivative of position, so acceleration is the time derivative of the velocity.

Average Acceleration

Since the average velocity is related to the position by $\overline{v} = \Delta x / \Delta t$ we can similarly write the average acceleration in terms of the velocity by

$$\overline{a} = \frac{\Delta v}{\Delta t}.$$

We can think of average acceleration graphically as the slope of the secant lines of a v vs. t graph.

Instantaneous Acceleration

The instantaneous acceleration (or just acceleration) is the time derivative of the velocity.

$$a = \frac{dv}{dt} = \dot{v} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

This can be written as the second derivative of the position.

$$a = \frac{d^2 x}{d t^2} = \ddot{x}$$

Derivatives of Polynomials

This section will be a self-contained discussion of just the material needed to evaluate derivatives of polynomials. The rules of differentiation needed for this are:

The Sum Rule

$$\frac{d}{d t}(u + v) = \frac{d u}{d t} + \frac{d v}{d t}$$

Constants

$$\frac{d}{d t} C = 0 \text{ and } \frac{d}{d t}(C u) = C \frac{d u}{d t} \text{ where } C \text{ is } a \text{ constant.}$$

Power Rule

$$\frac{d}{d t} t^p = p\, t^{p-1} \quad \left(\text{Note } \frac{d}{d t} t = 1\right)$$

Using the four rules above we can evaluate the derivative of any polynomial function. This is a straightforward thing. Consider an example:

$$\frac{d}{d t}\left(5 t^3 - 4 t^2 + 3 t + 8\right) = 5 \frac{d}{d t} t^3 - 4 \frac{d}{d t} t^2 + 3 \frac{d}{d t} t + \frac{d}{d t} 8 = 5 \cdot 3 t^2 - 4 \cdot 2 t + 3 \cdot 1 + 0 = 15 t^2 - 8 t + 3$$

Antiderivatives

The reverse of the differentiation procedure is called antidifferentiation. If $f(t)$ is a function then its antiderivative is another function $F(t)$ that satisfies $\frac{d}{dt} F(t) = f(t)$. Generally, antidifferentiation is a more complicated procedure than differentiation. For the simple case of polynomials, however, it is quite simple.

The basic idea is simple. Since the derivative of a constant is zero, if $F(t)$ is an antiderivative of $f(t)$ then $F(t) + C$ is also an antiderivative of $f(t)$, where C is an arbitrary constant. The result is stronger that this: $F(t) + C$ is the most general antiderivative of $f(t)$.

Note that an antiderivative of t^p is $\frac{1}{p+1} t^{p+1}$ for $p \neq -1$.

The Definite Integral

Since $v = d x / d t$ we can write the infinitesimal distance moved in the infinitesimal time $d t$ as

$$d x = v\, d t.$$

In a graph of v vs. t the area under the curve between t and $t + d t$ is $d x = v\, d t$; this is the area of a rectangle of height v and width $d t$. Thus the area under the graph over the infinitesimal time $d t$ is the infinitesimal distance travelled in that time. The total distance travelled between the times t_i and t_f is $\Delta x = x_f - x_i$; it is the sum of all the infinitesimal distances mentioned above, the total area under the curve. This is the definite integral of calculus.

$$\Delta x = \int_{t_i}^{t_f} v\, dt$$

Note the logic of this notation; in calculus d is implied to be a small Δ and the sum over an infinite number of infinitesimal things becomes the integral.

$$\Delta \Longrightarrow d \text{ and } \sum \Longrightarrow \int$$

The definite integral generally has the interpretation as the area under a curve. The area under $y = f(t)$ between a and b is written

$$\int_a^b f(t)\, dt$$

When the function is negative the contribution to the area is taken to be negative. Note how the velocity and position are related. $v = dx/dt$. The fundamental theorem of calculus is a generalization of this; it gives the rule we use to evaluate definite integrals.

$$\int_a^b f(t)\, dt = F(t)\,\Big|_a^b = F(b) - F(a) \text{ where } f(t) = \frac{d}{dt} F(t)$$

Thus, the definite integral of f is the difference of an antiderivative at the endpoints.

Constant Velocity and Acceleration

Now that we have considered the general problem of one dimensional kinematics we can now consider special cases. First constant velocity, then constant acceleration. An important case of constant acceleration is free fall.

Constant Velocity

If velocity is a constant then the acceleration is zero, since the derivative of a constant is zero. Let us now find the position from the velocity. Position is the antiderivative of the velocity.

$$\frac{dx}{dt} = v = \text{constant} \Longrightarrow x(t) = v\,t + C \text{ where } C \text{ is } a \text{ constant.}$$

Define the initial position x_0 to be the position at $t = 0$, $x_0 = x(0)$. Plugging this into our expression for $x(t)$ gives $C = x_0$ and

$$x(t) = x_0 + v\,t$$

If we choose the convention $t_i = 0, t_f = t, x_i = x_0$ and $x_f = x$ then we get $\Delta x = x - x_0$. The above expression becomes

$$x = x_0 + v\,t \text{ or } \Delta x = v\,t.$$

This is a simple expression; for constant velocity, the distance is the product of the velocity and time.

Constant Acceleration

If the acceleration is a constant then to get the velocity we repeat the procedure for going from a constant velocity to the position. Velocity is the antiderivative of the acceleration.

$$\frac{dv}{dt} = a = \text{constant} \Longrightarrow v(t) = a\,t + C_1 \text{ where } C_1 \text{ is } a \text{ constant.}$$

Define the initial velocity v_0 to be the velocity at $t = 0$, $v_0 = v(0)$. Plugging this into our expression for $v(t)$ gives $C_1 = v_0$ and

$$v(t) = v_0 + a\,t$$

We need to antidifferentiate again to get the position as a function of time.

$$\frac{dx}{dt} = v(t) = v_0 + a\,t \implies x(t) = v_0\,t + \frac{1}{2}\,a\,t^2 + C_2 \text{ where } C_2 \text{ is } a \text{ different constant.}$$

The arbitrary constant becomes the initial position x_0 and we get

$$x(t) = x_0 + v_0\,t + \frac{1}{2}\,a\,t^2.$$

If we choose the same conventions as in the constant velocity case and add $v_i = v_0$ and $v_f = v$ then the above expressions for velocity and position become

$$v = v_0 + a\,t \text{ and } \Delta\,x = v_0\,t + \frac{1}{2}\,a\,t^2.$$

Recall how to calculate Δx from v vs. t; it is the area under the curve between t_i and t_f. For constant acceleration the velocity vs. time is a straight line. We then get the area under a trapezoid with a base of width Δt and heights of v_i and v_f. This gives

$$\Delta x = \frac{1}{2}\left(v_i + v_f\right) \cdot \Delta t.$$

Using our convention for v_0 v and t, this becomes

$$\Delta x = \frac{1}{2}\left(v_0 + v\right)\,t.$$

We now want to derive an expression relating $\Delta\,x$, v_0, v and a. To do this that the previous expression and $v = v_0 + a\,t$, and then eliminate time.

$$\Delta x = \frac{1}{2}\left(v_0 + v\right)\frac{v - v_0}{a} \implies v^2 - v_0^2 = 2\,a\,\Delta x.$$

With this we have derived a set of four equations for kinematics with constant acceleration. These relate the variables t, $\Delta\,x$, v_0, v and a. These will be useful for a large class of problems this chapter.

Constant Acceleration Equations

$$v = v_0 + a\,t$$

$$\Delta x = \frac{1}{2}\left(v_0 + v\right)\,t$$

$$\Delta x = v_0\,t + \frac{1}{2}\,a\,t^2$$

$$v^2 - v_0^2 = 2\,a\,\Delta x$$

Free Fall

Free fall is one dimensional motion under the influence of only gravity. Assuming that only gravity acts implies that we are ignoring and friction effects. We will choose the convention that up is the positive direction. Also, we will take y as the position variable; this will be consistent with our later usage where y is typically taken as the vertical variable. Galileo discovered that the acceleration of all bodies in the presence of gravity (ignoring air resistance) is the same. The value of the downward acceleration is

$$g = 9.80\,\frac{\text{m}}{\text{s}^2} = 32.0\,\frac{\text{ft}}{\text{s}^2}.$$

Since up is the positive y direction and the acceleration is downward we take the acceleration to be:

$$a = -g$$

Using this value of a and replacing x with y takes the constant acceleration equations to the free fall expressions.

Free Fall Equations

$$v = v_0 - g\ t$$

$$\Delta y = \frac{1}{2}\left(v_0 + v\right)\ t$$

$$\Delta y = v_0\ t - \frac{1}{2}\ g\ t^2$$

$$v^2 - v_0^2 = -2\ g\ \Delta y$$