Chapter I - Problems

Blinn College - Physics 2425 - Terry Honan

Problem I.1

A 200 kg mass sits at $x = 0$ and a 500 kg mass sits at $x = 0.4$ m.
(a) What is the net gravitational force on a 50 kg mass at the midpoint of the two, $x = 0.2$ m.
(b) Where on the x axis would the net force on a third mass be zero.

Solution to I.1

$$m_1 = 200\,\text{kg} \quad \text{and} \quad m_2 = 500\,\text{kg}$$

(a) The force on $m = 50$ kg is the sum of the forces of m_1 and m_2 on m.

$$F_1 = G\,\frac{m_1\,m}{r_1^2} = 6.67\times 10^{-11}\,\frac{200\times 50}{0.2^2} = 1.6675\times 10^{-5}\,\text{N}$$

$$F_2 = G\,\frac{m_2\,m}{r_1^2} = 6.67\times 10^{-11}\,\frac{500\times 50}{0.2^2} = 4.1688\times 10^{-5}\,\text{N}$$

$\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2$. Since the forces are in opposite directions we should subtract their magnitudes.

$$F_{\text{net}} = F_2 - F_1 = 2.50\times 10^{-5}\,\text{N}$$

(b) Take the distances from m_1 and m_2 to be x and $x - d$, where $d = 0.4\,m$.

$$F_2 = F_1 \implies G\,\frac{m_1\,m}{x^2} = G\,\frac{m_2\,m}{(d-x)^2} \implies \frac{d-x}{x} = \sqrt{m_2/m_1} \implies d = x + x\sqrt{m_2/m_1}$$

$$\implies x = \frac{d}{1+\sqrt{m_2/m_1}} = \frac{0.4}{1+\sqrt{5/2}} = 0.155\,\text{m}$$

Problem I.2

Consider a 80 000 kg uniform solid sphere with a 1.2 m. What is the gravitational field a distance of r from the center for the values:
(a) $r = 0$, (b) $r = 0.6$ m, (c) $r = 1.2$ m, (d) $r = 2.4$ m

Solution to I.2

The gravitational field $\vec{g}$ is defined as the force per test mass, $\vec{g} = \vec{F}\big/m_0$. Here we are just looking for the magnitude of the field. The shell theorem implies the force and field at r go as

$$F = G\,\frac{M_{\text{inside}}\,m_0}{r^2} \implies g = \frac{F}{m_0} = G\,\frac{M_{\text{inside}}}{r^2}$$

where M_{inside} is the total mass inside a sphere of radius r.

Here take $M = 80\,000$ kg and $R = 1.2$ m. When outside the sphere M_{inside} is just M.

$$r \geq R \implies M_{\text{inside}} = M$$

When inside the sphere M_{inside} depends on the fraction of the total volume inside r: call this V_{inside}.

$$r < R \implies M_{\text{inside}} = M\,\frac{V_{\text{inside}}}{V_{\text{total}}} = M\,\frac{(4/3)\,\pi\,r^3}{(4/3)\,\pi\,R^3} = M\left(\frac{r}{R}\right)^3$$

(a) $r = 0 \implies M_{\text{inside}} = 0 \implies g = 0$

(b) $r = 0.6\text{ m} \implies M_{\text{inside}} = 80\,000\left(\frac{0.6}{1.2}\right)^3 = 10\,000 \implies g = G\,\frac{M_{\text{inside}}}{r^2} = 1.85\times10^{-6}\ \frac{\text{m}}{s^2}$

(c) $r = 1.2\text{ m} \implies M_{\text{inside}} = 80\,000 \implies g = G\,\frac{M_{\text{inside}}}{r^2} = 3.70\times10^{-6}\ \frac{\text{m}}{s^2}$

(d) $r = 2.4\text{ m} \implies M_{\text{inside}} = 80\,000 \implies g = G\,\frac{M_{\text{inside}}}{r^2} = 0.926\times10^{-6}\ \frac{\text{m}}{s^2}$

For the above calculations the value of $G = 6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$ was used.

Problem I.3

Four equal spherical masses m are put at the corners of a square is sides of length a. What is the total potential energy of the configuration?

Solution to I.3

The potential energy of a configuration is

$$U = -G \sum_{i<j} \frac{m_i\,m_j}{r_{ij}}.$$

We sum over all pairs. There are six pairs of four masses representing the four sides and the two diagonals. The distance of the diagonals is $\sqrt{2}\ a$.

$$U = -4\times G\,\frac{m^2}{a} - 2\times G\,\frac{m^2}{\sqrt{2}\ a} = -\frac{G\,m^2}{a}\left(4+\sqrt{2}\right)$$

Problem I.4

Planet X has 20 times the mass of the earth and 3 times the Earth's radius. It orbits star Y, which has 8 times the mass of the sun. The orbital radius of X is 5 AU, where 1 AU is the radius of the Earth's orbit about the sun.
(a) If an alien from planet X weighs 500 Zorgs on X then what is his Earth weight in Zorgs. (Zorgs are the alien unit of force and weight.)
(b) What is the orbital period of X about Y in years?

Solution to I.4

(a) The ratio of two weights is the same in any set of units. We can compare the weight on X to the Earth.

$$W = m\,g \implies \frac{W_X}{W_E} = \frac{g_X}{g_E}$$

We can also apply ratios to our foumula for g.

$$g = G\,\frac{M}{R^2} \implies \frac{g_X}{g_E} = \frac{M_X/M_E}{(R_X/R_E)^2}$$

$$\frac{W_X}{W_E} = \frac{M_X/M_E}{(R_X/R_E)^2} \implies \frac{500\text{ Zorgs}}{W_E} = \frac{20}{3^2} \implies W_E = 225\text{ Zorgs}$$

(b) The orbital period is found from $T^2 = \frac{4\pi^2}{G\,M}\,r^3$. Here M is the mass of the star and r is the radius of the orbit.

$$T^2 = \frac{4\pi^2}{G\,M}\,r^3 \implies \left(\frac{T_X}{T_E}\right)^2 = \frac{(r_X/r_E)^3}{M_Y/M_{\text{sun}}} \implies \left(\frac{T_X}{1\text{ yr}}\right)^2 = \frac{(5/1)^3}{8} \implies T_X = 3.95\text{ yr}$$

Problem I.5

The escape speed from the surface of a spherical planet is $2000\,m/s$. If a rocket is shot off this planet at a speed of $2500\,m/s$ then what is its speed when a long distance from the planet?

Solution to I.5

$$E = K + U = \frac{1}{2}\,m\,v^2 - \frac{G\,M\,m}{r}$$

If a mass begins with speed $v_0$at the surface of a planet of mass M and radius R, then its speed at infinity v∞ is found by

$$\frac{1}{2}\,m\,v_0^2 - \frac{G\,M\,m}{R} = \frac{1}{2}\,m\,v_\infty^2 - 0 \implies v_\infty = \sqrt{v_0^2 - \frac{2\,G\,M}{R}}$$

$$v_{\text{esc}} = \sqrt{\frac{2\,G\,M}{R}} \implies v_\infty = \sqrt{v_0^2 - v_{\text{esc}}^2}$$

Inserting the numbers gives

$$v_\infty = \sqrt{v_0^2 - v_{\text{esc}}^2} = \sqrt{2500^2 - 2000^2} = 1500\,\frac{\text{m}}{\text{s}}$$

Problem I.6

Two stars of equal mass M orbit about their common center of mass in a binary star system. Each star moves in a circular orbit of radius R so the distance between them is $2\,R$.

(a) Derive an expression for the speed v as a function of R and M.

(b) Eliminate R from the derived result by writing R in terms of the period T and speed, to obtain an expression relating v, T and M. Solve for the mass.

(c) Two stars in such a binary system are observed to orbit each other once every 18.3 days. Suppose also that Doppler shift data gives the speed of the masses to be $1.5\times10^5\,m/s$. What is the (equal) mass M of the stars? (Note: Binary star systems are important tools for finding masses of stars in astronomy.)

Solution to I.6

(a) Apply Newton's second law to one of the stars and use the usual expression for the centripetal acceleration.

$$F = m\,a_c \implies G\,\frac{M^2}{(2\,R)^2} = M\,\frac{v^2}{R} \implies v^2 = \frac{G\,M}{4\,R}$$

(b) For uniform circular motion we can write the radius in terms of the speed and radius. We can then eliminate R.

$$v = \frac{2\,\pi\,R}{T} \implies \frac{1}{R} = \frac{2\,\pi}{v\,T} \implies v^2 = \frac{G\,M}{4}\,\frac{1}{R} = \frac{G\,M}{4}\,\frac{2\,\pi}{v\,T} \implies v^3 = \frac{\pi\,G\,M}{2\,T}$$

The mass can now be found.

$$M = \frac{2\,T\,v^3}{\pi\,G}$$

(c) Given $T = 18.3$ days and $v = 1.5\times 10^5\ m/s$ we get:

$$M = \frac{2\,T\,v^3}{\pi\,G} = \frac{2\,(18.3\times 24\times 3600)\,(1.5\times 10^5)^3}{\pi\ 6.67\times 10^{-11}} = 5.09\times 10^{31}\ \text{kg}$$

Problem I.7

A satellite of mass m initially on the surface of a spherical planet of mass M and radius R. How much energy is needed to launch it into a circular orbit a height h above the planet.

Solution to I.7

The energy needed to get into orbit is $\Delta E = E_f - E_i$ where

$$E = K + U = \frac{1}{2}\,m\,v^2 - \frac{G\,M\,m}{r}.$$

At the surface it is at rest at $r_i = R$.

$$E_i = 0 - \frac{G\,M\,m}{R}.$$

In orbit at a height h the speed is

$$v^2 = \frac{G\,M}{r} = \frac{G\,M}{R+h}$$

The energy in orbit is

$$E_f = \frac{1}{2}\,m\,v^2 - \frac{G\,M\,m}{R+h} = \frac{1}{2}\,m\,\frac{G\,M}{R+h} - \frac{G\,M\,m}{R+h} = -\frac{1}{2}\,\frac{G\,M\,m}{R+h}.$$

It follows that the required energy is

$$\Delta E = E_f - E_i = -\frac{1}{2}\,\frac{G\,M\,m}{R+h} - -\frac{G\,M\,m}{R} = G\,M\,m\left(\frac{1}{R} - \frac{1}{2\,(R+h)}\right)$$

Problem I.8

(a) Consider any object orbitting the sun. Show that if the semimajor axis a is measured in AU, where 1 AU is the Earth-sun distance, and if the period T is measured in years then

$$T^2 = a^3 \quad (T \text{ in years and } a \text{ in AU})$$

(b) Halley's comet is observed to have a period of 75.6 yr. Its distance of closest approach is 0.57 AU. What is its largest distance from the sun in its orbit?

Solution to I.8

(a) Kepler's third law says that $T^2 \propto a^3$, where T is the period and a is the semimajor axis. a is half the largest distance between two points in an elliptical orbit; for circular orbits it is just the radius. We can rewrite the proportionality in terms of the earth's orbit, where $T =$ 1 year and $a =$ 1 AU.

$$\left(\frac{T}{T_{\text{earth}}}\right)^2 = \left(\frac{a}{a_{\text{earth}}}\right)^3 \implies T^2 = a^3,$$

where T is in years and a is in AU.

(b) For Halley's comet:

$$a = 75.6^{2/3} = 17.88 \text{ AU}.$$

If r_{min} and r_{max} are the smallest and largest distance of the comet from the sun then

$$r_{\text{min}} + r_{\text{max}} = 2a \implies r_{\text{max}} = 2a - r_{\text{min}} = 2 \times 17.88 - 0.57 = 35.2 \text{ AU}$$

Problem I.9

Suppose a spherical planet has no atmosphere or one of negligeable thickness.
(a) Show that the escape speed is $\sqrt{2}$ larger than the speed of an orbit just above the surface.
(b) A satellite orbits with a period T just above the surface. What is the average density of the planet? Density is the mass per volume.

Solution to I.9

(a) The speed of a circular orbit is $v = \sqrt{GM/r}$. Just above the surface of a planet $r = R$. The escape speed is $v_{\text{esc}} = \sqrt{2GM/R}$. Thus $v_{\text{esc}} = \sqrt{2}\, v$.

(b) The period of a satellite in a circular orbit is

$$T^2 = \frac{4\pi^2}{GM} r^3.$$

The density ρ is the mass per volume

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi r^3} \implies M = \rho \frac{4}{3}\pi r^3 \implies T^2 = \frac{4\pi^2 r^3}{G\rho \frac{4}{3}\pi r^3} = \frac{3\pi}{G\rho} \implies \rho = \frac{3\pi}{G T^2}$$