Chapter 8: Conservation of Linear Momentum

Linear Momentum

Definition: The linear momentum $\vec{p}$ of an object having mass m moving with velocity $\vec{v}$ is defined to be:

$$\vec{p} \equiv m\vec{v} \quad (1)$$

Note:

- Unit: $\mathrm{kg} \cdot \dfrac{\mathrm{m}}{\mathrm{s}}$
- Momentum is a **vector quantity.** In general, for an object moving in 3-D, (1) becomes

$$\langle p_x, p_y, p_z \rangle = m \langle v_x, v_y, v_z \rangle \quad (2)$$

  and this really implies *three* equations:

$$p_x = mv_x \quad (3)$$

$$p_y = mv_y \quad (4)$$

$$p_z = mv_z \quad (5)$$

➢ $\vec{p}$ is important in part because sometimes it's conserved!

Conservation of Linear Momentum

When (i.e., under what condition(s)) is $\vec{p}$ conserved? The answer comes from Newton's 2nd law in its *original* form. Originally, Newton wrote the 2nd law in terms of what we would today recognize as momentum. (Newton called it "the quantity of motion.")

$$\sum \vec{F} = \frac{d\vec{p}}{dt} \quad (\textit{Principia}, 1687) \tag{6}$$

For the special case in which the mass of the object is not changing, (6) is equivalent to the more familiar form of the 2nd law:

$$\sum \vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(m\vec{v}\right)$$

$$\sum \vec{F} = m\frac{d\vec{v}}{dt}$$

$$\sum \vec{F} = m\vec{a} \tag{7}$$

Eq. (6) is thus more general than (7).

Eq. (6) also contains within it the condition that is required for $\vec{p}$ to be conserved. If $\vec{p}$ is conserved, then

$$\frac{d\vec{p}}{dt} = \vec{0}$$

and from (6), this implies

$$\sum \vec{F} = \vec{0} \qquad (8)$$

This observation is known as the law of conservation of linear momentum, and it is one of the most important statements in the entire course:

Law of Conservation of Linear Momentum: The linear momentum of an object (or system) is conserved if the net external force on it is zero.

One famous example of a situation in which $\vec{p}$ is conserved is that of a *collision*. Consider, for example, a collision between two billiard balls on a pool table. Assume any external forces on the system of the two balls (friction, e.g.) are small enough to be ignored. Then applying Newton's 2^{nd} law (in its *original* form) to Ball #1, I get:

$$\sum \vec{F}_{\text{on Ball \#1}} = \frac{d\vec{p}_1}{dt}$$

$$\vec{F}_{21} = \frac{d\vec{p}_1}{dt}, \qquad (*)$$

in which $\vec{F}_{21}$ means the contact force that Ball #2 exerts on Ball #1. Doing the same for Ball #2, I get:

$$\sum \vec{F}_{\text{on Ball \#2}} = \frac{d\vec{p}_2}{dt}$$

$$\vec{F}_{12} = \frac{d\vec{p}_2}{dt} \qquad (**)$$

By Newton's 3^{rd} law, $\vec{F}_{12} = -\vec{F}_{21}$, so $(**)$ becomes

$$\vec{F}_{21} = -\frac{d\vec{p}_2}{dt} \qquad (***)$$

Equations ($*$) and ($***$) are two different equations for the same quantity, $\vec{F}_{21}$. Equating the right-hand sides, I get:

$$\frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt},$$

and integrating on both sides:

$$\int_{t_i}^{t_f} \left(\frac{d\vec{p}_1}{dt} \right) dt = -\int_{t_i}^{t_f} \left(\frac{d\vec{p}_2}{dt} \right) dt$$

or, using the Fundamental Theorem of Calculus:

$$\Delta\vec{p}_1 = -\Delta\vec{p}_2 \qquad (9)$$

This says that if Ball #2 *gains* a certain amount of momentum, Ball #1 must *lose* an *equal* amount of momentum; therefore, the total momentum of the system stays the *same* (i.e., it is **conserved**)! More formally, we can write the statement of conservation of linear momentum for this collision by explicitly writing out $\Delta\vec{p}_1$ and $\Delta\vec{p}_2$ in (9), then gathering terms:

$$m_1\vec{v}_{1f} - m_1\vec{v}_{1i} = -\left(m_2\vec{v}_{2f} - m_2\vec{v}_{2i} \right)$$

$$m_1\vec{v}_{1f} + m_2\vec{v}_{2f} = m_1\vec{v}_{1i} + m_2\vec{v}_{2i} \qquad (10)$$

Collisions

- Two broad categories:

  1. Head-on ("1-D"): entire collision takes place along single line (x axis)
  2. Glancing ("2-D"): entire collision does **not** take place along single line.

- Within each of these 2 categories, 3 types:

  1. Elastic
  2. Inelastic
  3. Perfectly inelastic (or "completely inelastic", or "totally inelastic")

➢ What distinguishes the three types from one another is what's conserved.

Elastic

- Total $\vec{p}$ of system conserved
- Total K of system conserved
- no permanent deformation of objects (car bumper at low speed, rubber ball, rubber-band bumper on air-track gliders, billiard balls, etc.)

Inelastic

- Total $\vec{p}$ of system conserved
- Total K of system **not** conserved
- There **is** some permanent deformation of one or both objects
  - some K "lost" in process of deforming one or both objects; don't get this energy back after the collision
  - car accident at "high" speed, e.g.

Perfectly Inelastic

- is inelastic, so $\vec{p}$ conserved, K not
- **special case:** objects *stick together* (railroad cars, e.g.)

Head-on (1-D) Collisions

- Entire collision takes place along single line
- Can call this line the x axis; then can write:

$$\vec{v}_{1i} = (v_{1i})\hat{i}$$
$$\vec{v}_{2i} = (v_{2i})\hat{i}$$
$$\vec{v}_{1f} = (v_{1f})\hat{i}$$
$$\vec{v}_{2f} = (v_{2f})\hat{i}$$

- $\vec{p}$-conservation then becomes:

$$(m_1 v_{1i} + m_2 v_{2i})\hat{i} = (m_1 v_{1f} + m_2 v_{2f})\hat{i},$$

which implies:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \qquad (11)$$

So get to drop the $\hat{i}$ unit vectors. **But remember:** each of the v's means the component of that velocity in the x direction; this component can be + or -.

Perfectly Inelastic

- just know $\vec{p}$ conserved, so can write down *only*:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

  except $v_{1f} = v_{2f} = v_f$, so $\vec{p}$-conservation becomes:

$$m_1 v_{1i} + m_2 v_{2i} = \left(m_1 + m_2\right) v_f \tag{12}$$

- can't write down any statement about K being conserved, because it is *not* conserved!

- So have just one equation; can solve for just one unknown!

Elastic

- Know *two* things that are conserved: $\vec{p}$ and K. So:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \tag{13}$$

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \tag{14}$$

- In principle, just solve the pair of simultaneous equations represented by (13) and (14) for whatever the problem asked for (v_{1f} and v_{2f}, e.g.). But **messy!** So can use (13) and (14) to derive a third formula that involves just the *velocities themselves* and **not** *their squares*. Rewrite (13) as:

$$m_1 \left(v_{1i} - v_{1f} \right) = m_2 \left(v_{2f} - v_{2i} \right) \tag{$*$}$$

In similar way, rewrite (14) as:

$$m_1 \left(v_{1i}^2 - v_{1f}^2 \right) = m_2 \left(v_{2f}^2 - v_{2i}^2 \right) \tag{$**$}$$

Now divide ($**$) by ($*$) to get:

$$v_{1i} + v_{1f} = v_{2i} + v_{2f} \tag{15}$$

- Can use (13) and (15) to solve for v_{1f} and v_{2f} *instead* of using (13) and (14). Less messy!

Inelastic (but not *perfectly* inelastic)

- Only $\vec{p}$ conserved. So can write down only:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \quad (16)$$

- Have just one equation. Therefore, can solve for just one unknown.

Glancing (2-D) Collisions

- Collision does **not** place along single line.
- Total $\vec{p}$ of system still conserved, but can't write $\vec{p}$-conservation as:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} ,$$

with no arrows over the v's. So what to do?

- Back up a step… $\vec{p}$ is a **vector**, so $\vec{p}$-conservation says:

$$\vec{p}_i = \vec{p}_f ,$$

in which $\vec{p}_i$ and $\vec{p}_f$ have to be understood to mean the initial and final momenta of the **entire system.** But:

$$\vec{p}_i = \left\langle p_{ix}, p_{iy} \right\rangle$$

and:

$$\vec{p}_f = \left\langle p_{fx}, p_{fy} \right\rangle$$

So if $\vec{p}$ *conserved*, then must have:

$$p_{ix} = p_{fx} \qquad (17)$$

and:

$$p_{iy} = p_{fy} \qquad (18)$$

➢ **start with (17) and (18) for glancing!**

Impulse

Newton's 2nd law says:

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}$$

$$\vec{F}_{net}\,dt = d\vec{p}$$

Integrating on both sides:

$$\int_{t_i}^{t_f} \vec{F}_{net}\,dt = \int_{\vec{p}_i}^{\vec{p}_f} d\vec{p}$$

$$\int_{t_i}^{t_f} \vec{F}_{net}\,dt = \Delta\vec{p} \qquad (19)$$

The quantity on the LHS of (19), the integral of a force over some time interval, is called the impulse, $\vec{J}$:

$$\vec{J} \equiv \int_{t_i}^{t_f} \vec{F}(t)\,dt \qquad (20)$$

The Impulse-Momentum Theorem

With the definition of impulse in (20), we can write (19) as:

$$\vec{J}_{net} = \Delta\vec{p} \qquad (21)$$

This is called the impulse-momentum theorem. It says that the net impulse exerted on some object equals the change in that object's momentum.

The impulse-momentum theorem is really just a restatement of Newton's 2nd law, but it is useful for situations (e.g., *collisions*) in which the details of how the force varies with time are not well known. The impulse-momentum theorem lets us get *some* information about the *average* force just by knowing (or *measuring*) the initial and final momenta. Notice that the definition of $\vec{J}$ given in (20) looks almost like the *average force*, $\vec{F}_{av}$. In fact, it follows from (20) that

$$\left(\vec{F}_{net}\right)_{av} = \frac{\vec{J}_{net}}{\Delta t} \qquad (22)$$

or:

$$\left(\vec{F}_{net}\right)_{av} = \frac{\Delta\vec{p}}{\Delta t} \qquad (23)$$

Center of Mass and Systems of Particles

Consider a system of N point particles located at points in the x-y plane:

$$(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$$

with masses

$$m_1, m_2, \ldots, m_N,$$

respectively.

The coordinates of the center of mass (CM) of this system of particles are then defined to be:

$$X_{CM} \equiv \frac{m_1 x_1 + m_2 x_2 + \cdots + m_N x_N}{m_1 + m_2 + \cdots + m_N} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_N x_N}{M_{tot}} \quad (24)$$

$$Y_{CM} \equiv \frac{m_1 y_1 + m_2 y_2 + \cdots + m_N y_N}{m_1 + m_2 + \cdots + m_N} = \frac{m_1 y_1 + m_2 y_2 + \cdots + m_N y_N}{M_{tot}} \quad (25)$$

- The coordinates of the CM are weighted averages of the coordinates of all the particles (weighted by the mass at each location)!

The position vector of the CM is given by

$$\vec{r}_{CM} = \left\langle X_{CM}, Y_{CM} \right\rangle = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N}{M_{tot}} \qquad (26)$$

To see this, write out the position vectors $\vec{r}_1 = \left\langle x_1, y_1 \right\rangle$ through $\vec{r}_N = \left\langle x_N, y_N \right\rangle$ in (26):

$$\vec{r}_{CM} = \left\langle X_{CM}, Y_{CM} \right\rangle = \frac{m_1\left\langle x_1, y_1 \right\rangle + m_2\left\langle x_2, y_2 \right\rangle + \cdots + m_N\left\langle x_N, y_N \right\rangle}{M_{tot}}$$

Now distribute the masses on the RHS. Then add all the x components and (separately) all the y components:

$$\vec{r}_{CM} = \left\langle X_{CM}, Y_{CM} \right\rangle = \left\langle \frac{m_1x_1 + m_2x_2 + \cdots + m_Nx_N}{M_{tot}}, \frac{m_1y_1 + m_2y_2 + \cdots + m_Ny_N}{M_{tot}} \right\rangle$$

Thus, (26) is equivalent to (24) and (25).

To see the significance of the CM, differentiate (26) twice with respect to time:

$$\frac{d^2\vec{r}_{CM}}{dt^2} = \frac{m_1\left(\dfrac{d^2\vec{r}_1}{dt^2}\right) + m_2\left(\dfrac{d^2\vec{r}_2}{dt^2}\right) + \cdots + m_N\left(\dfrac{d^2\vec{r}_N}{dt^2}\right)}{M_{tot}}$$

$$M_{tot}\vec{a}_{CM} = m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_N\vec{a}_N \qquad (*)$$

By Newton's 2nd law, each "$m\vec{a}$" term on the RHS of ($*$) equals the net force on that particle. Therefore, the entire RHS equals the net force on the entire system (vector sum of all the forces on all the particles). So:

$$\sum\vec{F} = M_{tot}\vec{a}_{CM} \qquad (27)$$

Eq. (27) says that the entire system of particles acts as though all the mass were *concentrated* at the CM, in the sense that the acceleration of the CM will be the same as if the entire system of particles were replaced with a *single* particle of mass M_{tot} located at the CM.

It is for this reason that, in FBDs, your book shows forces acting at the *centers* of the objects. If the object is of uniform density, then the center of mass of the object is *at the geometrical center* of the object.

Note here that the LHS of (27) includes, strictly speaking, *all* forces on *all* particles, whether these forces are *internal* or *external*. But we can always partition the net force into two pieces and write (27) as:

$$\sum \vec{F}_{int} + \sum \vec{F}_{ext} = M_{tot} \vec{a}_{CM}$$

The sum of all the *internal* forces is *zero* because the internal forces always cancel pairwise due to Newton's 3rd law. So we have:

$$\sum \vec{F}_{ext} = M_{tot} \vec{a}_{CM} \qquad (28)$$

which, since the mass is assumed constant, is equivalent to:

$$\sum \vec{F}_{ext} = \frac{d\vec{P}}{dt}, \qquad (29)$$

where $\vec{P}$ is the total momentum of the system.

Here's the significance of (28) and (29). These two equations say that the only thing that can accelerate the center of mass (or, equivalently, change the momentum of the entire system) is a *net external force.* Any *internal* force can do *nothing* to change the motion of the CM (or, equivalently, to change the momentum of the entire system).