Chapter 7: Conservation of Energy

Energy

Comes in various forms:

- chemical (energy stored in a battery, e.g.)
- electrical (energy associated w/flow of electric current, e.g.)
- mechanical (energy of mechanical systems)
    - could be simple, such as block sliding down incline, Atwood's machine, etc.
    - or could be more complex, such as internal combustion engine.

➢ Will focus only on *mechanical* energy in this course.

➢ Energy can be *transformed* (from one *form* to another) or *transferred* (from one *object* to another) but the total amount of energy in the Universe stays **fixed** (law of conservation of energy).

Potential Energy

Within mechanical energy, 2 types:

1. kinetic energy, K: energy that *moving* objects have because of their motion (i.e., because they are moving)

$$K \equiv \frac{1}{2}mv^2 \tag{1}$$

   - Note that only *moving* object have kinetic energy

2. potential energy, U: energy that an object can have because of its *position* (relative to some **reference level:** place where the potential energy is chosen to be *zero*).

The total mechanical energy, E, is the sum of the kinetic energy and all forms of potential energy:

$$E \equiv K + U \tag{2}$$

We will deal with 2 types of potential energy in this course:

1. gravitational potential energy, U_{grav}: energy that an object can have because of its position *near the surface of the Earth*.

$$U_{grav} \equiv mgy \tag{3}$$

   - valid *only* near surface!
   - m is the mass of the object
   - $g = 9.81\ \mathrm{m/s^2}$ (acceleration due to gravity)
   - y is the *position* of the object relative to some chosen reference level that *could* be the surface of the Earth, but doesn't *have* to be.
   - Note the unit for U_{grav}: $\mathrm{kg} \cdot \frac{\mathrm{m}}{\mathrm{s}^2} \cdot \mathrm{m} = \mathrm{J}$

2. elastic potential energy, U_{el}: potential energy stored in a deformation (of a *spring*, or of an *area*, or of a *volume*, e.g.) More about this later.

Conservative and Nonconservative Forces

For some forces, called conservative forces, it is always possible to write the work done by the force as $-\Delta U$, for some potential energy function U.

These kinds of forces are called *conservative* because they *conserve* the total mechanical energy (i.e., they leave the total mechanical energy *unchanged*).

There are only two conservative forces we will deal with in this course:

1. the gravitational force, $F_{grav} = mg$
2. the spring force, $F_{spring} = kx$

For nonconservative forces, it is **not** possible to write the work done by the force as $-\Delta U$. These forces **do** change the total mechanical energy, **if** they do any work at all. Examples include friction, normal forces, tensions, and any kind of *driving force*.

To see that the gravitational force is a conservative force, consider any object that moves along an arbitrary path from some initial point $P_i\left(x_i, y_i, z_i\right)$ to some final point $P_f\left(x_f, y_f, z_f\right)$. (Let the $+y$ axis be vertically upward.) The work done by the gravitational force is, from the definition of work:

$$W_{grav} = \int \vec{F}_{grav} \cdot d\vec{r}$$

$$W_{grav} = \int_{y_i}^{y_f} \left(-mg\right) dy$$

If we're near the surface of the Earth, g is approximately *constant*, so (assuming the mass of the object is not changing):

$$W_{grav} = \left(-mg\right) \int_{y_i}^{y_f} dy = -mg\left(y_f - y_i\right) \qquad (4)$$

Now consider the change in potential energy of the object:

$$\Delta U_{grav} = U_{grav}^{f} - U_{grav}^{i} = mgy_f - mgy_i = mg\left(y_f - y_i\right) \qquad (5)$$

Comparing (4) and (5), we see that $W_{grav} = -\Delta U_{grav}$. Therefore, the gravitational force is conservative.

Properties of Conservative Forces

1. The work done can be written $W = -\Delta U$.
2. The work done between any two points is *independent of the path* between the points.
3. For any *closed* path, the work done is *zero*.

The Law of Conservation of Energy

As applied to the mechanical energy of an individual object or a *system* of objects, the law of conservation of energy says if the only forces doing work on a system are *conservative* forces, then the total mechanical energy of the system is conserved.

This is one of the most important statements in the entire course.

- Note that the energy can still be conserved if there are nonconservative forces *acting* on the object or system; the nonconservative forces just can't be doing any *work* on the object or system.

The fact that the total mechanical energy is conserved if the only forces doing work are conservative ones follows from the fact that, for conservative forces, it's possible to write the work done as $-\Delta U$, for some potential energy, U. If the *only* forces doing work are *conservative* forces, then the *net* work done can be written:

$$W_{net} = -\Delta U \tag{6}$$

But, by the work-energy theorem, we also know:

$$W_{net} = \Delta K \tag{7}$$

Combining (6) and (7) gives

$$\Delta K = -\Delta U \tag{8}$$

This says that the total mechanical energy E remains unchanged. (If the object or system *loses* a certain amount of *potential* energy, it *gains* an *equal* amount of *kinetic* energy.) More formally, we can write (8) as:

$$K_f - K_i = -\left(U_f - U_i\right)$$

$$K_f + U_f = K_i + U_i$$

$$E_f = E_i \tag{9}$$

- Note here that the potential energy U must be understood to include *all* forms of potential energy (*gravitational* and *elastic*, in this course).

The Reference Level for the Gravitational Potential Energy

- The level at which $U_{grav} = mgy$ is defined to be *zero*. In other words, the level at which y is defined to be *zero*.
- You can always choose the reference level to be wherever you want.

➢ Frequently useful trick: choose the reference level so as to make either U^{i}_{grav} or U^{f}_{grav} zero.

The Elastic Potential Energy

We said earlier that the spring force is a conservative force. Therefore, it must be possible to write the work done by the spring force as:

$$W_{spring} = -\Delta U_{el}, \qquad (10)$$

for some *elastic potential energy*, U_{el}. We would now like to find the potential energy function that satisfies (10).

To do so, consider a horizontal spring with its left end attached to a vertical wall and the free end (the right end) initially at $x = 0$. Imagine someone grabbing the free end of the spring and pulling it to the right *quasi-statically* with some applied force $\vec{F}_{applied}$. (We imagine doing this *quasi-statically* to avoid increasing the spring's *kinetic energy*.) By Hooke's law, as the spring is stretched, the applied force will have to grow in magnitude:

$$\vec{F}_{applied} = (kx)\hat{i}$$

in which k is the *force constant* of the spring.

By Newton’s 3rd law, the force that the *spring* exerts on the person’s hand is, then,

$$\vec{F}_{spring} = (-kx)\hat{i}$$

Because F_{spring} is not constant, we must integrate to find the work done by this force:

$$W_{spring} = \int_{x_i}^{x_f} \left(F_{spring}\right) dx = \int_{x_i}^{x_f} (-kx)\, dx = -\frac{1}{2}kx_f^2 - \left(-\frac{1}{2}kx_i^2\right)$$

$$W_{spring} = -\left[\frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2\right] \qquad (11)$$

Comparing (11) and (10), it seems natural to identify the first term in square brackets in (11) as U_{el}^{f} and the second term as U_{el}^{i}:

$$U_{el}^{f} = \frac{1}{2}kx_f^2$$

$$U_{el}^{i} = \frac{1}{2}kx_i^2$$

or, for any x:

$$U_{el} \equiv \frac{1}{2}kx^2 \qquad (12)$$

Then (11) becomes

$$W_{spring} = -\left(U_{el}^{f} - U_{el}^{i}\right) = -\Delta U_{el}$$

Thus, the definition of elastic potential energy chosen in (12) guarantees that the spring force will be a conservative force.

- Note the choice of “reference level” in (12): U_{el} is chosen to be zero when $x = 0$ (i.e., when the spring is at its equilibrium length). This seems like a natural choice, but it is not the only one. We could have chosen the reference level so that U_{el} differed from the expression in (12) by any *constant* and it would still be true that $W_{spring} = -\Delta U_{el}$.

When doing conservation of energy problems involving springs, then, we just have to include $U_{el} = (1/2)kx^2$ when we write down the initial and final energies.

Work Done by Nonconservative Forces

What if there are nonconservative forces doing work? Then the energy is not conserved. In fact, consider the work-energy theorem:

$$W_{net} = \Delta K$$

The net work can always be thought of as being made up of two parts: the work done by all the *conservative* forces *plus* the work done by all the *nonconservative* ones. So:

$$W_c + W_{nc} = \Delta K$$

But

$$W_c = -\Delta U$$

so:

$$-\Delta U + W_{nc} = \Delta K$$

$$W_{nc} = \Delta K + \Delta U$$

$$W_{nc} = \Delta E \qquad (13)$$

When nonconservative forces **do** some work, the energy changes. Furthermore, the work done by all the nonconservative forces **equals** the amount by which the energy changes.

Force and Potential Energy

Consider a varying force in 1-D, $\vec{F} = \langle F_x(x), 0, 0 \rangle$. If this force is conservative, then the work done by the force is:

$$W = -\Delta U$$

or

$$\int_{x_i}^{x_f} F_x \, dx = -\Delta U$$

or (rearranging ever so slightly)

$$\int_{x_i}^{x_f} (-F_x) \, dx = \Delta U$$

From the Fundamental Theorem of Calculus, it follows that the function U is an antiderivative of $-F_x$:

$$-F_x = \frac{dU}{dx}$$

or

$$F_x = -\frac{dU}{dx} \qquad (14)$$

In general, the potential energy could depend on x, y, and z:

$$U = U\left(x, y, z\right)$$

In this case, F_x is the negative of the *partial* derivative of U with respect to x:

$$F_x = -\frac{\partial U}{\partial x}$$

The *partial derivative* with respect to x is the derivative with respect to x, treating all other variables as *constants*.

In the most general case, the force could have x, y, and z components, each of which depends on all three spatial variables:

$$\vec{F} = \left\langle F_x\left(x, y, z\right), F_y\left(x, y, z\right), F_z\left(x, y, z\right) \right\rangle$$

In this case, we would have:

$$F_x = -\frac{\partial U}{\partial x} \tag{15}$$

$$F_y = -\frac{\partial U}{\partial y} \tag{16}$$

$$F_z = -\frac{\partial U}{\partial z} \tag{17}$$

Energy Diagrams

- Graphs of U vs. x.
- Wherever there is a *local maximum* or a *local minimum*, $dU/dx = 0$ and, by (14), $F_x = 0 \Rightarrow$ object is in **equilibrium!**
  - o stable equilibria: graph is *concave up*; any displacement away from equilibrium gives rise to a force that drives the system *back toward* equilibrium.

$$\frac{d^2U}{dx^2} > 0 \text{, stable equilibria} \tag{18}$$

  - o unstable equilibria: graph is *concave down*; any displacement away from equilibrium gives rise to a force that drives the system *farther away from* equilibrium.

$$\frac{d^2U}{dx^2} < 0 \text{, unstable equilibria} \tag{19}$$