Chapter 3: Kinematics in 2-D and 3-D

Recall in our discussion of 1-D kinematics, we said that there are three key concepts when describing the motion of an object: the displacement, the velocity, and the acceleration. These were defined for motion in 1-D as follows:

Displacement: $\Delta x \equiv x_f - x_i$

Velocity: $v_{av} \equiv \dfrac{\Delta x}{\Delta t}$ (avg.) and $v \equiv \dfrac{dx}{dt}$ (instantaneous)

Acceleration: $a_{av} \equiv \dfrac{\Delta v}{\Delta t}$ (avg.) and $a \equiv \dfrac{dv}{dt}$ (instantaneous)

Motion in two dimensions means motion in a plane. Motion in three dimensions means motion through some volume of space. For motion in 2-D and 3-D, the above definitions need to be modified.

Displacement, Velocity and Acceleration in 2-D

Before we can define the displacement for things that move in 2-D, we need to decide how to describe the *position*.

The way we described the position for things that move in 1-D was to simply give a *single* real number representing the position along the x axis, relative to some chosen origin. This definition is inadequate for things that move in 2-D.

Position in 2-D

The position of an object moving in 2-D is represented by a *vector* drawn from the origin of the coordinate system to the location of the object. This is called the position vector of the object, $\vec{r}$. For an object located at point (x, y), then, the position vector would be:

$$\vec{r} = \langle x, y \rangle \qquad (1)$$

Displacement in 2-D

The displacement in 2-D is defined as follows:

$$\Delta\vec{r} \equiv \vec{r}_f - \vec{r}_i = \left\langle x_f, y_f \right\rangle - \left\langle x_i, y_i \right\rangle = \left\langle \Delta x, \Delta y \right\rangle \qquad (2)$$

Note how similar this is in form to the definition of the displacement for things that move in 1-D, $\Delta x = x_f - x_i$. The only difference is that the position is now a vector, so the displacement is also a vector.

Velocity in 2-D

The velocity for things that move in 2-D is also a vector. There are *two kinds* of velocity, just as for motion in 1-D:

Average:

$$\vec{v}_{av} \equiv \frac{\Delta\vec{r}}{\Delta t} = \left\langle \frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t} \right\rangle \qquad (3)$$

Instantaneous:

$$\vec{v} \equiv \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\langle x, y \rangle = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \langle v_x, v_y \rangle \qquad (4)$$

Notice once again how similar these definitions are to the corresponding definitions for 1-D motion.

Also Note:

- The time interval Δt in Eqs. (3) and (4) is *never negative*.
- As a result, $\vec{v}_{av}$ **always points in the same direction as** $\Delta\vec{r}$ **.**
- $\vec{v}$ **always points tangent to the path, in the direction of motion.**

Acceleration in 2-D

The acceleration for things that move in 2-D is also a vector. There are *two kinds* of acceleration, just as for motion in 1-D:

Average:

$$\vec{a}_{av} \equiv \frac{\Delta\vec{v}}{\Delta t} = \left\langle \frac{\Delta v_x}{\Delta t}, \frac{\Delta v_y}{\Delta t} \right\rangle \tag{5}$$

Instantaneous:

$$\vec{a} \equiv \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left\langle v_x, v_y \right\rangle = \left\langle \frac{dv_x}{dt}, \frac{dv_y}{dt} \right\rangle = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle = \frac{d^2\vec{r}}{dt^2} \tag{6}$$

Notice once again how similar these definitions are to the corresponding definitions for 1-D motion.

Parallel and Perpendicular Components of Acceleration

Considering the definition of $\vec{a}_{av}$ in Eq. (5), we see that **there is some acceleration (i.e., $\vec{a}_{av} \neq \vec{0}$) whenever $\vec{v}$ is changing in any way.**

There are two ways the velocity vector can change:

- $\vec{v}$ can change in **magnitude.**
- $\vec{v}$ can change in **direction.**

In either case, there is some acceleration.

In general, then, the acceleration vector can have a tangential component $a_{||}$ (tangent to the path) and a radial component $a_{\perp}$ (perpendicular to the path).

The **tangential component** changes the **magnitude** of the velocity vector but **cannot** cause a change in the **direction** of the velocity vector. Thus, $a_{||}$ represents how rapidly the magnitude of the velocity is changing:

$$a_{||} \equiv \frac{d\left|\vec{v}\right|}{dt} \qquad (7)$$

The **radial component** changes the **direction** of $\vec{v}$ but **cannot** cause a change in the **magnitude** of $\vec{v}$. The magnitude of this component of the acceleration vector is related to the speed $v = \left|\vec{v}\right|$ and to the radius of curvature of the path, r, by:

$$a_{\perp} \equiv \frac{v^2}{r} \qquad (8)$$

This component always points **centripetally** (i.e., toward the center of curvature).

In general, then, the total acceleration vector $\vec{a}$ is the vector sum of the tangential and radial components:

$$\vec{a} = \vec{a}_{\parallel} + \vec{a}_{\perp} \quad (9)$$

Because $\vec{a}_{\parallel}$ and $\vec{a}_{\perp}$ are *necessarily* at right angles to one another, the magnitude of the total acceleration can be found from the Pythagorean Theorem:

$$|\vec{a}| = \sqrt{a_{\parallel}^2 + a_{\perp}^2} \quad (10)$$

Displacement, Velocity and Acceleration in 3-D

Just as in 2-D, the position, displacement, velocity and acceleration for things that move along paths in 3-D are vectors.

The only difference is that there is now an extra term in each definition.

Position in 3-D
The position is once again described by a position vector $\vec{r}$ from the origin to the location of the object. Thus, for an object at (x, y, z):

$$\vec{r} = \langle x, y, z \rangle \tag{11}$$

Displacement in 3-D
The displacement in 3-D is defined as follows:

$$\Delta\vec{r} \equiv \vec{r}_f - \vec{r}_i = \left\langle x_f, y_f, z_f \right\rangle - \left\langle x_i, y_i, z_i \right\rangle = \left\langle \Delta x, \Delta y, \Delta z \right\rangle \tag{12}$$

Velocity in 3-D

Average:

$$\vec{v}_{av} \equiv \frac{\Delta \vec{r}}{\Delta t} = \left\langle \frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t} \right\rangle \quad (13)$$

Instantaneous:

$$\vec{v} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\langle x, y, z \rangle = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = \langle v_x, v_y, v_z \rangle \quad (14)$$

Acceleration in 3-D

Average:

$$\vec{a}_{av} \equiv \frac{\Delta \vec{v}}{\Delta t} = \left\langle \frac{\Delta v_x}{\Delta t}, \frac{\Delta v_y}{\Delta t}, \frac{\Delta v_z}{\Delta t} \right\rangle \quad (15)$$

Instantaneous:

$$\vec{a} \equiv \frac{d\vec{v}}{dt} = \frac{d}{dt}\langle v_x, v_y, v_z \rangle = \left\langle \frac{dv_x}{dt}, \frac{dv_y}{dt}, \frac{dv_z}{dt} \right\rangle = \left\langle \frac{d^2 x}{dt^2}, \frac{d^2 y}{dt^2}, \frac{d^2 z}{dt^2} \right\rangle = \frac{d^2 \vec{r}}{dt^2} \quad (16)$$

Projectile Motion

Projectile motion is an application of 2-D kinematics.

A projectile is an object that is given an initial velocity (by *some means*) and which goes into **free-fall** upon being released.

The x and y parts of the motion are separable.

X Part:

$$a_x = 0 \tag{17}$$

$$v_x = v_{0x} \tag{18}$$

$$x = x_0 + v_{0x}t \tag{19}$$

Y Part: just free-fall!

$$v_y = v_{0y} - gt \tag{20}$$

$$y = y_0 + \frac{1}{2}\left(v_{0y} + v_y\right)t \tag{21}$$

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2 \tag{22}$$

$$v_y^2 = v_{0y}^2 - 2g\left(y - y_0\right) \tag{23}$$

Relative Velocity

Consider an object *A* moving in 3-D. The position vector of this object relative to the origin *B* of the "unprimed" coordinate system (or reference frame) is:

$$\vec{r}_{AB} = \langle x, y, z \rangle$$

The position vector of the same point *A* relative to the origin *C* of the "primed" reference frame is:

$$\vec{r}_{AC} = \langle x', y', z' \rangle$$

These two position vectors are related to one another by:

$$\vec{r}_{AB} = \vec{r}_{AC} + \vec{r}_{CB} \qquad \textbf{(24)}$$

Differentiating (24) with respect to time, I get:

$$\frac{d\vec{r}_{AB}}{dt} = \frac{d\vec{r}_{AC}}{dt} + \frac{d\vec{r}_{CB}}{dt}$$

Or:

$$\vec{v}_{AB} = \vec{v}_{AC} + \vec{v}_{CB} \qquad \textbf{(25)}$$