

# Chapter 13: Periodic Motion (Oscillations)

**Oscillation:** periodic “back-and-forth” motion between two extremes (or *limits*).

- requires a **restoring force**: force that restores the system to its equilibrium position after you displace it *away from* equilibrium.

We will focus on **simple harmonic motion (SHM)**. In SHM, the restoring force just depends **linearly** on the displacement away from equilibrium.

## Simple Harmonic Motion (SHM)

Consider a glider connected to one end of a spring, oscillating back and forth on an air track, as shown in Figure 1.

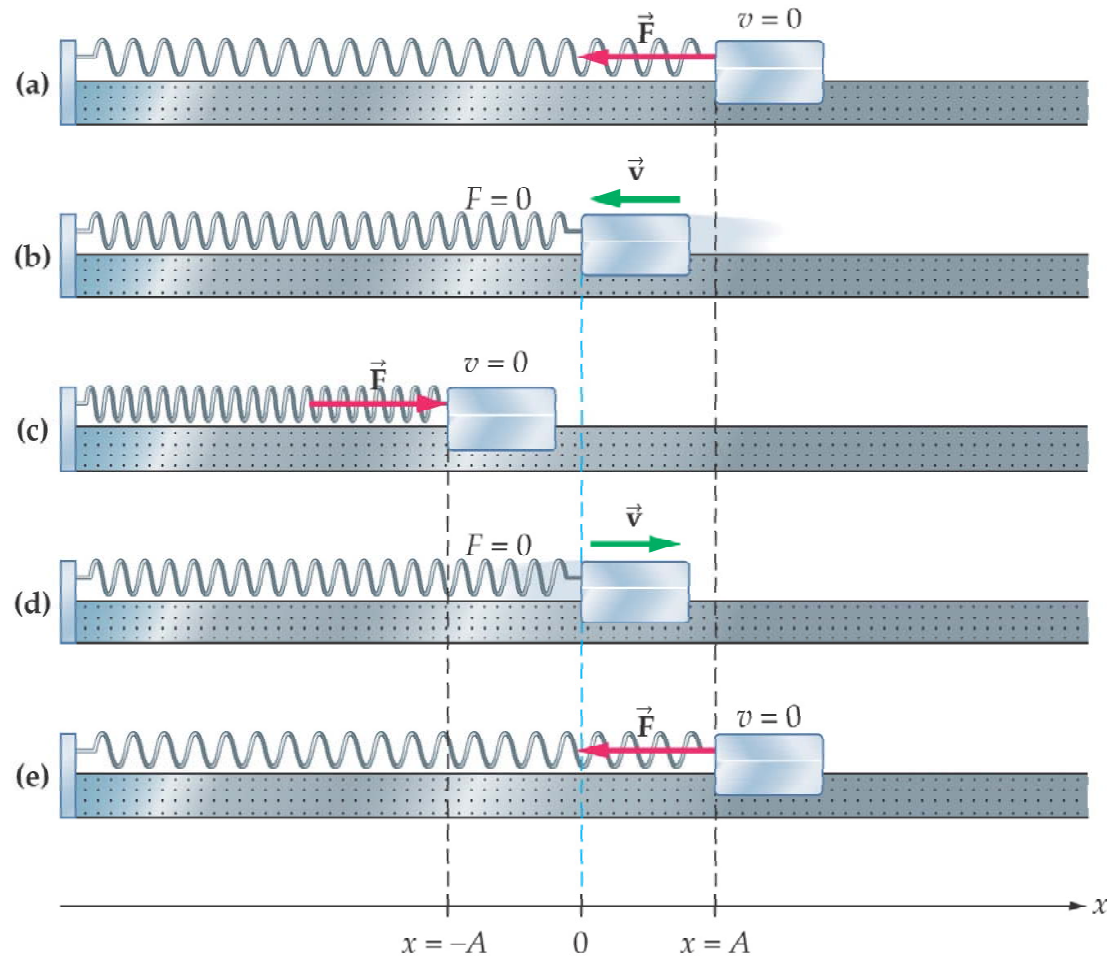


Figure 1

As it oscillates back and forth, the position of the glider along the  $x$  axis is described by a **sinusoid** (a sine or cosine), as shown in Figure 2.

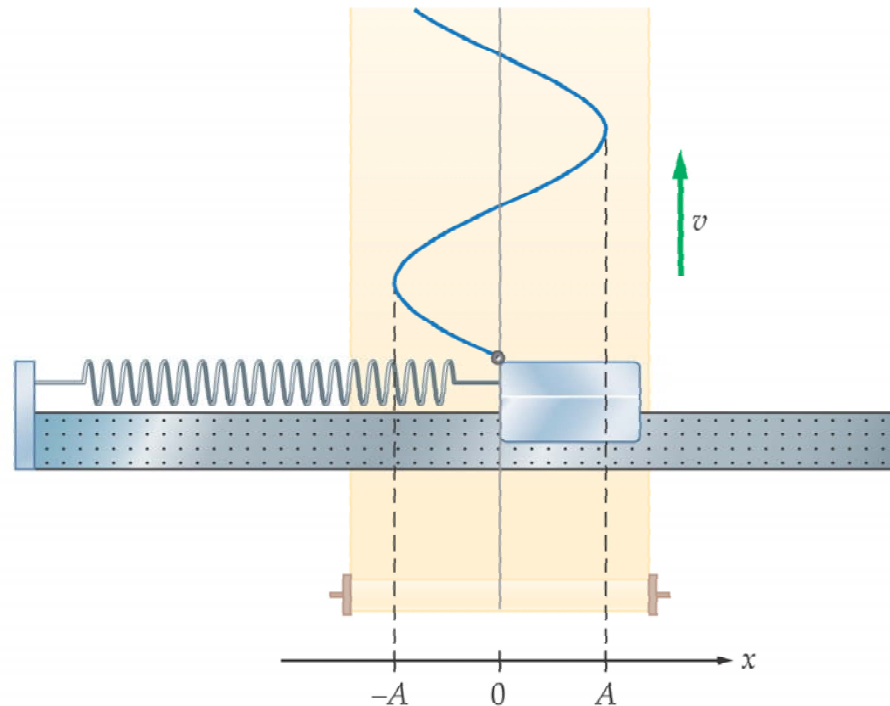


Figure 2

The most general sinusoid is  $A \cos(\omega t + \phi)$ . Accordingly, I will assume that  $x(t)$  is given by

$$x(t) = A \cos(\omega t + \phi) \quad (1)$$

## Properties of Sinusoids

Consider the graph of  $x(t) = A \cos \omega t$  shown in Figure 3.

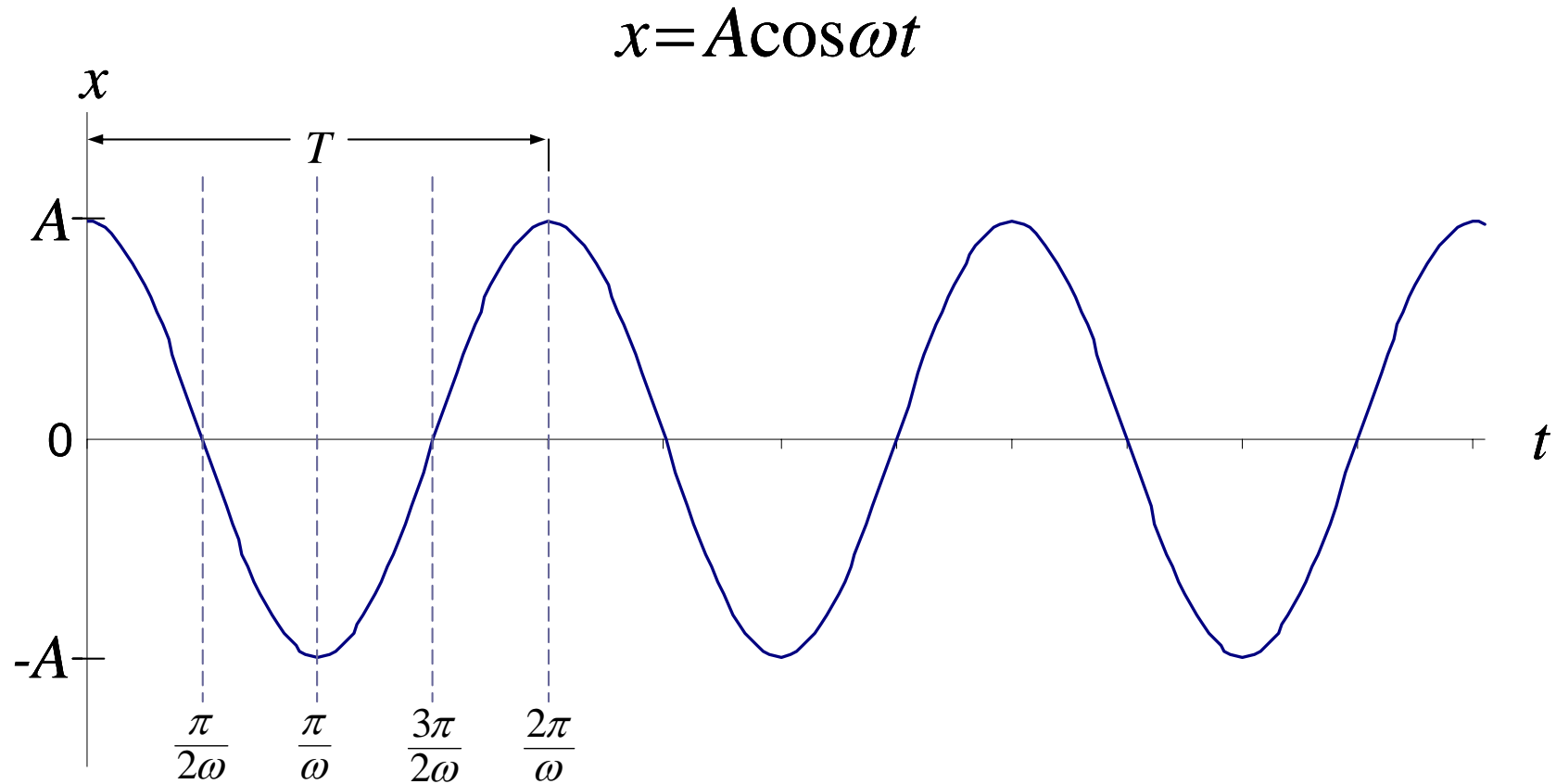


Figure 3

*Key Terms:*

- The **period**,  $T$ , is the time required for one complete cycle.
- The **amplitude**,  $A$ , is the magnitude of the maximum displacement away from zero.
- The **frequency**,  $f$ , is the number of complete cycles per second.
- The **angular frequency**,  $\omega$ , is the number of radians that the angle  $\omega t$  goes through per second.

The frequency is related to the period by

$$f = \frac{1}{T} \quad (2)$$

The unit for  $f$  is cycles/s ( $s^{-1}$ ). This is called a *hertz* in honor of Heinrich Hertz.

The angular frequency is related to the frequency by

$$\omega = 2\pi f \quad (3)$$

The unit for  $\omega$  is rad/s.

It follows that the period is related to the angular frequency by

$$T = \frac{2\pi}{\omega} \quad (4)$$

The **phase angle**,  $\phi$ , sets the starting point of the wave, as shown in Figure 4.

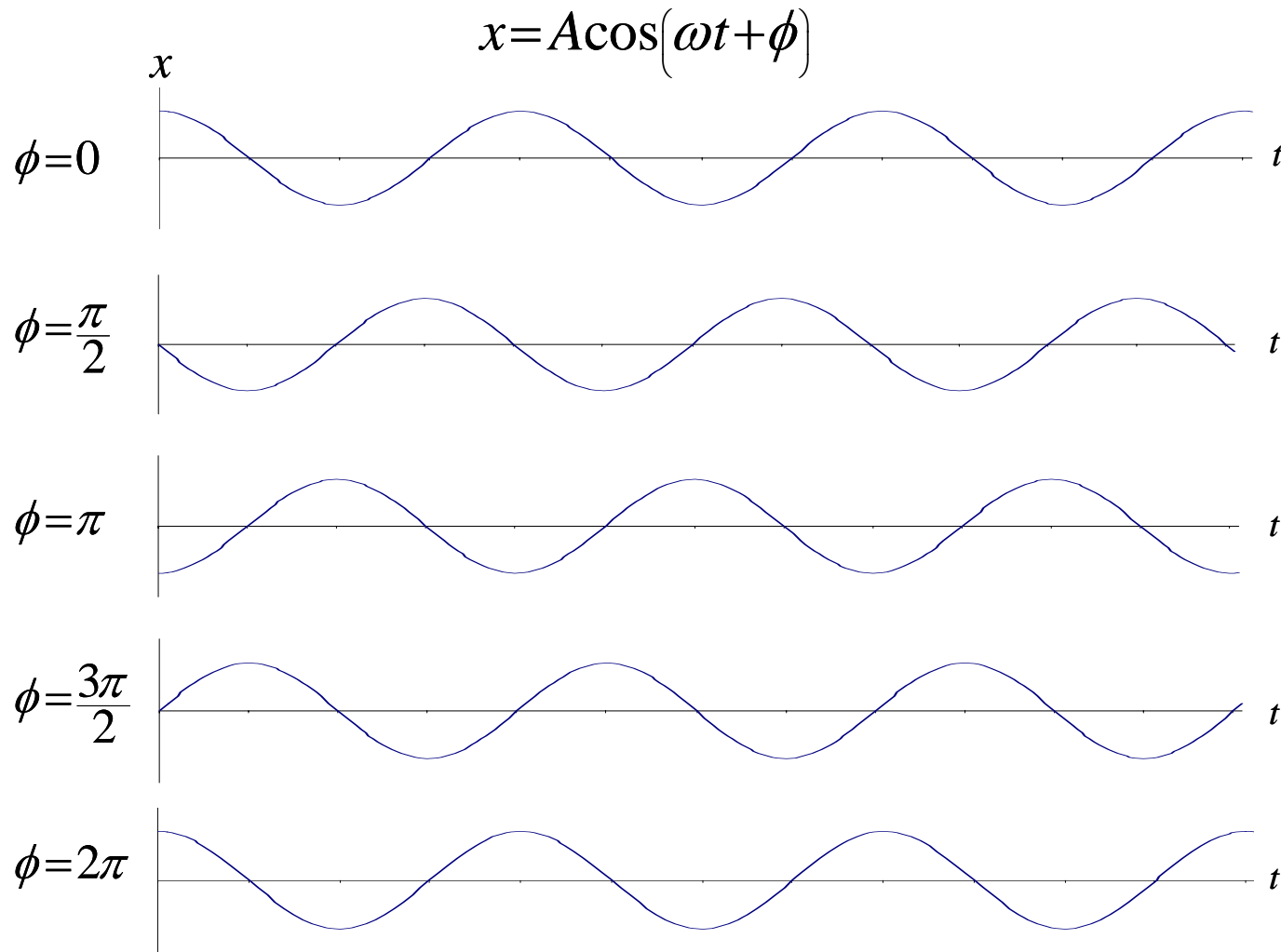


Figure 4

## Connection Between Oscillatory Motion and Uniform Circular Motion

Oscillatory motion is the **projection** of uniform circular motion onto a **line** (the  $x$  axis or  $y$  axis, for example).

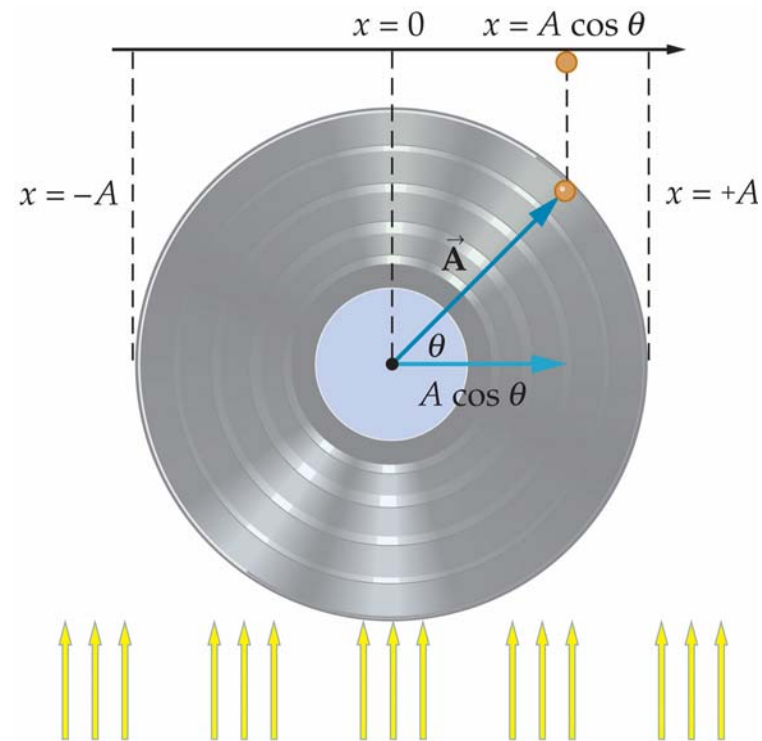


Figure 5

## Period of a Mass on a Spring

Newton's second law, applied to the glider in Figure 1, says

$$\begin{aligned}\sum F_x &= m \frac{d^2 x}{dt^2} \\ -kx &= m \frac{d^2 x}{dt^2}\end{aligned}\tag{5}$$

Note the minus sign in the Hooke's-law force. For *positive* values of  $x$ , the force must point in the  $-x$  direction; for *negative* values of  $x$ , the force must point in the  $+x$  direction.

Eq. (5) is a **second-order ordinary differential equation** for the function  $x(t)$ . A **differential equation**, recall, is an equation involving a function and its derivatives. The **order** of a differential equation refers to the highest derivative that appears in the equation. The highest derivative appearing in (5) is the *second* derivative, so this is a second-order differential equation.

Eq. (5) is called an **ordinary** differential equation because it involves just *ordinary* derivatives (*total* derivatives) of  $x(t)$ , not *partial* derivatives.

**Solving** a differential equation means finding the function that satisfies it. On the basis of the earlier discussion, I will *guess* that the solution is of the form  $x(t) = A \cos(\omega t + \phi)$ .

The constants  $A$  and  $\phi$  are determined by **initial conditions**. The constant  $\omega$  depends on physical parameters of the system, as we will discover.

Differentiating  $x(t)$  twice with respect to time, I get:

$$\frac{dx}{dt} = v(t) = -A \sin(\omega t + \phi) \frac{d}{dt}[\omega t + \phi] = -\omega A \sin(\omega t + \phi) \quad (6)$$

$$\frac{d^2x}{dt^2} = a(t) = -\omega A \cos(\omega t + \phi) \frac{d}{dt}[\omega t + \phi] = -\omega^2 A \cos(\omega t + \phi) = -\omega^2 x \quad (7)$$

Notice that I had to use the **chain rule** in (6) and (7).

Plugging (7) back into (5), I get

$$\begin{aligned} -kx &= m(-\omega^2 x) \\ \omega &= \sqrt{\frac{k}{m}} \end{aligned} \tag{8}$$

So  $x(t) = A\cos(\omega t + \phi)$  is a solution to (5), and the constant  $\omega$  is related to the force constant of the spring and the mass of the glider by (8).

The period of the motion now follows at once from (4):

$$\begin{aligned} T &= \frac{2\pi}{\sqrt{k/m}} \\ T &= 2\pi\sqrt{\frac{m}{k}} \end{aligned} \tag{9}$$

**Notes:**

- $T$  is long (large) for large masses.
- $T$  is short (small) for large  $k$  (stiff springs).
- $T$  does **not** depend on the amplitude  $A$ !

## Determining the Constants $A$ and $\phi$ from Initial Conditions

At  $t = 0$ , the position is  $x(0) = x_0$ . Plugging  $t = 0$  into (1), I get:

$$x_0 = A \cos \phi \quad (10)$$

At  $t = 0$ , the *velocity* is  $v(0) = v_0$ . Plugging  $t = 0$  into (6), I get:

$$v_0 = -\omega A \sin \phi \quad (11)$$

Doing a little algebra with (10) and (11), I find:

$$A^2 (\cos^2 \phi + \sin^2 \phi) = x_0^2 + \left(\frac{v_0}{\omega}\right)^2$$

But  $\cos^2 \phi + \sin^2 \phi = 1$ , so:

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} \quad (12)$$

To find  $\phi$  in terms of initial conditions, I divide (11) by (10) to get:

$$\tan \phi = -\frac{v_0}{\omega x_0} \Rightarrow \phi = \tan^{-1} \left( -\frac{v_0}{\omega x_0} \right) \quad (13)$$

If the object starts *from rest*,  $v_0 = 0$ , so:

$$A = x_0 \quad (14)$$

and

$$\phi = \tan^{-1}(0) = 0 \quad (15)$$

## Conservation of Energy (revisited)

If the work done by the friction force is negligibly small, the only force doing any work on the glider in Figure 1 is the spring force, which is a conservative force. Therefore, the total mechanical energy is conserved:

$$\begin{aligned} E_f &= E_i \\ K_f + U_{el}^f &= K_i + U_{el}^i \\ \frac{1}{2}mv_f^2 + \frac{1}{2}kx_f^2 &= \frac{1}{2}mv_i^2 + \frac{1}{2}kx_i^2 \end{aligned} \quad (16)$$

Let one of the positions ( $x_i$ , for example) correspond to a **turning point**: a place where the glider changes direction. Then  $x_i = A$  and  $v_i = 0$ . (The glider is momentarily *at rest*, so all the energy is stored in the spring.)

So:

$$\frac{1}{2}mv_f^2 + \frac{1}{2}kx_f^2 = \frac{1}{2}kA^2 \quad (17)$$

Because the total energy is conserved,  $E$  must equal  $(1/2)kA^2$  at all times. Therefore, the  $v_f$  and  $x_f$  in (17) can be taken to mean the velocity and position, respectively, at *any* point in the motion.

## Vertical SHM

Consider a mass hung from a *vertical* spring (of negligible mass). When the mass reaches equilibrium, the spring will be *stretched* (away from its *relaxed* length) by an amount  $\Delta\ell$ . Thus, Newton's 2<sup>nd</sup> law gives:

$$\begin{aligned}\sum F_y &= 0 \\ k\Delta\ell - mg &= 0 \\ k\Delta\ell &= mg\end{aligned}\quad (*)$$

Let us define this equilibrium position of the mass to be  $y = 0$  and take the positive  $y$  direction to be upward. Now assume the mass is displaced upward to a new position  $y$ . The spring will now be stretched by an amount  $\Delta\ell - y$ . Therefore, the net force on the mass will be:

$$\sum F_y = k(\Delta\ell - y) - mg,$$

but, using (\*), this becomes:

$$\sum F_y = -ky \quad (18)$$

Similarly, if the mass is displaced downward, the spring will be stretched by an amount  $\Delta\ell + |y| = \Delta\ell - y$  (since  $y$  is *negative*). So we once again get (18). Recalling (5), we see that (18) has the same form, but with  $x$  replaced by  $y$ . Therefore, the mass will undergo SHM about  $y = 0$ .

## The Simple Pendulum

A **simple pendulum** is a *point mass* suspended by a string (or rod, etc.) of *negligible* mass.

The motion of a simple pendulum is another example of SHM (for *small* oscillations of the pendulum.)

Consider displacing the pendulum “bob” (the point mass) away from the vertical by an angle  $\theta$ . The restoring force is then the tangential component of the weight:

$$F_{\parallel} = -mg \sin \theta \quad (*)$$

Note the minus sign in (\*). For *positive* values of  $\theta$ ,  $F_{\parallel}$  tends to make  $\theta$  *smaller*, so  $F_{\parallel}$  should be *negative*; for *negative* values of  $\theta$ ,  $F_{\parallel}$  tends to make  $\theta$  *larger* (less negative), so  $F_{\parallel}$  should be *positive*.

For *small* oscillations,  $\theta$  is small, and

$$\sin \theta \approx \theta \quad (\text{small-angle approximation}) \quad (19)$$

So (\*) becomes:

$$F_{\parallel} = -mg\theta$$

But from the definition of  $\theta$  in *radians*:

$$\theta = \frac{s}{r} = \frac{s}{L}$$

(See figure.)

Therefore:

$$F_{\parallel} = -\left(\frac{mg}{L}\right)s \quad (**)$$

Eq. (\*\*\*) has the same form as the restoring force in Eq. (5): the restoring force is minus some constant times the first power of the displacement away from equilibrium. Therefore, we could re-do all of our earlier analysis to get the period  $T$ , with the result that we would get an expression looking exactly like Eq. (9), but with the replacement:

$$k \rightarrow \frac{mg}{L}$$

Therefore, the period of the simple pendulum is given by:

$$T = 2\pi \sqrt{\frac{m}{(mg/L)}}$$
$$T = 2\pi \sqrt{\frac{L}{g}} \quad (20)$$

**Notes:**

- $T$  is long (large) for long strings and short (small) for short strings.
- $T$  does **not** depend on the mass!
- $T$  does **not** depend on the amplitude  $\theta_0$ !

The foregoing analysis is correct only for *small* oscillations (small amplitudes  $\theta_0$ ). If  $\theta_0$  is *large*, the correct expression for the period is an infinite series:

$$T = 2\pi \sqrt{\frac{L}{g}} \left( 1 + \frac{1^2}{2^2} \sin^2 \frac{\theta_0}{2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \sin^4 \frac{\theta_0}{2} + \dots \right) \quad (21)$$

You will use this equation to explore the period for large-angle oscillations in the lab.