

Chapter 12: Gravitation

Earlier in the course, we talked about the fact that the net gravitational force on an object is called the *weight* of the object, and we wrote this as $F_{grav} = w = mg$, in which $g = 9.81 \text{ m/s}^2$ **near the surface of the Earth.**

Also, we discussed gravitational potential energy, $U_{grav} \equiv mgy$, **near the surface of the Earth.**

In this chapter, we treat the subject of gravitation in a little more detail.

In particular, we focus on how the above expressions for the gravitational force and the gravitational potential energy must be corrected if we are **not** near the surface of the Earth.

Newton's Law of Universal Gravitation

Newton's law of universal gravitation, first published in 1687 in *Principia*, says that any two point masses anywhere in the Universe attract each other with a force that is directly proportional to the product of the masses and inversely proportional to the square of the distance between them:

$$F_{grav} = G \frac{m_1 m_2}{r^2} \quad (1)$$

The constant of proportionality, G , is called the **gravitational constant**, and is determined by experiment. It was first successfully measured by Sir Henry Cavendish in 1798. The modern accepted value is about

$$G = 6.671 \times 10^{-11} \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2} \quad (2)$$

Note the smallness of this number. The gravitational force becomes appreciable only for very large masses!

Note also the universality of this law: Newton says **any** two masses, anywhere in the Universe, attract one another with the force given in (1).

As given in (1), Newton's law of universal gravitation is a statement about the gravitational force of attraction between two **point masses**. It turns out that, for **spherically symmetric distributions of mass**, the same equation describes the force of attraction, except that the distance r must be taken to mean the distance between the **centers** of the objects. It was to prove this result that Newton had to invent the entire branch of mathematics we know today as integral calculus.

Weight

We are now ready to discover how the expression for the weight of an object must be modified if the object is not near the surface of the Earth.

The **weight** of an object is the net gravitational force exerted on the object. The *correct* expression for this force, valid at *any* r , is given by (1). Suppose the object we're talking about is, for example, a projectile of mass m moving above the Earth. Then, ignoring the gravitational force on the projectile due to bodies other than the Earth (such as the Moon):

$$F_{grav} = G \frac{mM_E}{r^2}, \quad (*)$$

in which M_E is the mass of the Earth and r is the center-to-center distance between the projectile and Earth.

But we also know that

$$F_{grav} = mg, \quad (**)$$

in which g is the acceleration due to gravity. So we have two different expressions for F_{grav} . Equating them, I get:

$$mg = G \frac{mM_E}{r^2}$$
$$g = G \frac{M_E}{r^2} \quad (3)$$

Thus, we can express the gravitational force at any distance r from the center of the Earth as $w = mg$, but g **is no longer constant!** It drops off like $1/r^2$ as we get farther and farther from the Earth.

At the surface of the Earth, $r \approx R_E$, and (3) becomes:

$$g_{surf} = G \frac{M_E}{R_E^2} = \frac{\left(6.671 \times 10^{-11} \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2}\right) (5.97 \times 10^{24} \text{ kg})}{(6.38 \times 10^6 \text{ m})^2} \approx 9.81 \text{ m/s}^2$$

To get the value of g at the surface of any other planet, moon, asteroid, etc., we just replace M_E and R_E in the expression immediately above with the mass and radius of that planet, moon, etc:

$$g_{surf} = G \frac{M}{R^2} \quad (4)$$

Gravitational Potential Energy

Earlier in the course, we said

$$U_{grav} \equiv mgy,$$

but this is valid only near the surface of the Earth. What if we're talking about an object that's **not** near the surface of the Earth? How should the expression for U_{grav} be modified?

Consider the work done by some external agent in raising a mass m from r_i to r_f (measured relative to the center of the Earth). Let $F_{applied}$ be the magnitude of the force that the external agent exerts on m . To avoid changing the kinetic energy of the mass m , let us assume, furthermore, that m is raised *quasi-statically*. Then:

$$F_{applied} \approx F_{grav} = G \frac{mM_E}{r^2}$$

This is a variable force. To calculate the work it does, we must do an integral:

$$W_{applied} = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{applied} \cdot d\vec{r}$$

Along a radial path from r_i to r_f , $\vec{F}_{\text{applied}} = (GmM_E / r^2) \hat{r}$ and $d\vec{r} = dr \hat{r}$, so:

$$W_{\text{applied}} = \int_{\vec{r}_i}^{\vec{r}_f} \left[\left(G \frac{mM_E}{r^2} \right) \hat{r} \right] \cdot [dr \hat{r}] = GmM_E \int_{r_i}^{r_f} \frac{1}{r^2} (\hat{r} \cdot \hat{r}) dr$$

But $\hat{r} \cdot \hat{r} = 1$, so

$$W_{\text{applied}} = GmM_E \int_{r_i}^{r_f} \frac{1}{r^2} dr = -GmM_E \left[\frac{1}{r} \right]_{r_i}^{r_f} = \left(-G \frac{mM_E}{r_f} \right) - \left(-G \frac{mM_E}{r_i} \right) \quad (*)$$

All of this work goes into increasing the gravitational potential energy, so:

$$W_{\text{applied}} = \Delta U_{\text{grav}} = U_{\text{grav}}^f - U_{\text{grav}}^i \quad (**)$$

Comparing (*) and (**), it seems natural to identify U_{grav}^f and U_{grav}^i as:

$$U_{\text{grav}}^f = -G \frac{mM_E}{r_f}$$

$$U_{\text{grav}}^i = -G \frac{mM_E}{r_i}$$

For any mass m at a distance r from the center of the Earth, we therefore define U_{grav} to be:

$$U_{grav} \equiv -G \frac{mM_E}{r}$$

More generally, for any two masses m_1 and m_2 separated by center-to-center distance r :

$$U_{grav} \equiv -G \frac{m_1 m_2}{r} \tag{5}$$

Conservation of Energy (revisited)

The gravitational force is a conservative force. Therefore, if the only force doing any work is gravitational force, the total mechanical energy will be conserved. This means:

$$\begin{aligned} E_f &= E_i \\ K_f + U_{grav}^f &= K_i + U_{grav}^i \end{aligned} \quad (6)$$

We will not concern ourselves with any elastic potential energy U_{el} here. Also, we will assume that any rotational kinetic energy is much smaller than the translational kinetic energy, so the K in (6) means K_{trans} .

For a projectile of mass m moving in the vicinity of the Earth, then, (6) becomes:

$$\frac{1}{2}mv_f^2 - G\frac{mM_E}{r_f} = \frac{1}{2}mv_i^2 - G\frac{mM_E}{r_i} \quad (7)$$

Escape Speed

The **escape speed** of a projectile is the minimum launch speed required in order for the projectile to leave and **never come back**.

Consider a projectile of mass m launched from a body (planet, moon, etc.) of mass M and radius R . Ignoring air resistance, once the projectile is launched, the only force acting on it is the gravitational force. Therefore:

$$E_i = E_f$$
$$\frac{1}{2}mv_i^2 - G\frac{mM}{r_i} = \frac{1}{2}mv_f^2 - G\frac{mM}{r_f} \quad (*)$$

The projectile is launched from the *surface*, so $r_i \approx R$.

As the projectile rises, it gains gravitational potential energy and loses kinetic energy, slowing down eventually to a stop. In order for the projectile never to come back, F_{grav} must go to **zero** when the projectile stops. Recall the expression for F_{grav} :

$$F_{grav} = G \frac{mM}{r^2}$$

This goes to zero as $r \rightarrow \infty$.

If we launch the projectile with a speed equal to the escape speed, then, it will go to $r = \infty$ and **stop**. Thus, $K_f = U_{grav}^f = 0$, and (*) becomes:

$$\frac{1}{2}mv_{esc}^2 - G \frac{mM}{R} = 0$$

Solving for v_{esc} gives:

$$v_{esc} = \sqrt{\frac{2GM}{R}} \quad (8)$$

Kepler's Laws

- Describe the motion of planets about the Sun, moons about planets, etc.
- Discovered by the German astronomer and mathematician Johannes Kepler between 1601 and 1619.
- Kepler did not know *why* the planets moved this way; Newton explained why in the *Principia* in 1687. All of Kepler's laws can be derived from Newton's laws of motion and Newton's law of universal gravitation.

Kepler's laws are as follows:

1. Each planet moves in an elliptical orbit with the Sun at one focus.
2. A line from the Sun to the planet sweeps out equal areas in equal times.
3. The period of the orbit of each planet is proportional to the $3/2$ power of its semi-major axis.

Kepler's 2nd Law

- follows from conservation of angular momentum
- To see this, consider the path of a planet around the Sun from time t to time $t + dt$.

The position vector \vec{r} and $\vec{v} dt$ form a *parallelogram*. The area of this parallelogram is

$$|\vec{r} \times \vec{v} dt|$$

The area swept out by a line from the Sun to the planet is $\frac{1}{2}$ this area. This infinitesimal area swept out in the time dt is therefore:

$$dA = \frac{1}{2} |\vec{r} \times \vec{v} dt| = \frac{1}{2m} |\vec{r} \times m\vec{v}| dt = \frac{1}{2m} |\vec{r} \times \vec{p}| dt = \left(\frac{L}{2m} \right) dt \quad (*)$$

Because the net force acting on the planet (the gravitational force) is **central** (i.e., radial, directed toward the Sun):

$$\vec{\tau}_{net} = \vec{r} \times \vec{F}_{net} = \vec{0}$$

Therefore, by the rotational form of Newton's 2nd law:

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt} = \vec{0},$$

which says that the angular momentum is **conserved**. So:

$$\frac{L}{2m} = \text{constant}$$

Eq. (*) therefore says:

$$dA = (\text{constant}) dt$$

Integrating on both sides from any t_i to $t_f = t_i + \Delta t$, I get:

$$\Delta A = \int_{t_i}^{t_f} dA = (\text{constant}) \int_{t_i}^{t_f} dt$$
$$\Delta A = (\text{constant}) \Delta t$$

This says that in equal time intervals Δt , equal areas ΔA will be swept out. This is Kepler's 2nd law.

Kepler's 3rd Law

Kepler's 3rd law is a little complicated to show for the case of an elliptical orbit. But for the special case of a *circular* orbit, it's much easier.

Consider a planet of mass M_p in a circular orbit about the Sun. (The orbits of many planets in our solar system, including the orbit of the Earth, are very nearly circular.) Newton's 2nd law, applied to the radial (centripetal) direction, says:

$$\begin{aligned}\sum F_{rad} &= M_p a_{rad} \\ G \frac{M_p M_s}{r^2} &= M_p \left(\frac{v^2}{r} \right) \\ \frac{GM_s}{r} &= v^2, \quad (*)\end{aligned}$$

in which M_s is the mass of the Sun and r is the radius of the orbit.

The orbital speed v is related to the orbital period T by:

$$v = \frac{\text{dist}}{\text{time}} = \frac{2\pi r}{T} \quad (**)$$

Plugging (**) into (*) and solving for T , I get:

$$\frac{GM_s}{r} = \left(\frac{2\pi r}{T} \right)^2$$
$$T = \frac{2\pi}{\sqrt{GM_s}} r^{3/2} \quad (9)$$

This is Kepler's 3rd law. Newton was able to show that the same relation holds for elliptical orbits if the radius r is replaced by the length of the semi-major axis, a :

$$T = \frac{2\pi}{\sqrt{GM_s}} a^{3/2} \quad (10)$$