

Chapter 10: Rotational Dynamics

Recall, said **dynamics** is the study of motion *and* its *causes*.

Torques cause changes in the rotational motion of an object.

Torque can be defined in terms of a **cross product**, so I'll first discuss the cross product (Sec. 1.10), then come back to the subject of torque.

Cross Product (“Vector Product”)

The **cross product** (“vector product”) of two vectors \vec{A} and \vec{B} is another vector (really a *pseudovector*) and is written $\vec{A} \times \vec{B}$. The magnitude of $\vec{A} \times \vec{B}$ is defined as follows:

$$|\vec{A} \times \vec{B}| \equiv |\vec{A}| |\vec{B}| \sin \phi = AB \sin \phi, \quad (1)$$

in which ϕ is the smallest angle between \vec{A} and \vec{B} ($0^\circ < \phi < 180^\circ$).

The direction of $\vec{A} \times \vec{B}$ is given by a “**right-hand rule (RHR)**”:

1. Point the fingers of right hand in direction of \vec{A} .
2. Curl fingers toward \vec{B} .
3. Thumb points in direction of $\vec{A} \times \vec{B}$.

➤ Note: $\vec{A} \times \vec{B}$ is always \perp to both \vec{A} and \vec{B} !

Properties of the Cross Product

1. The cross product is **anticommutative**:

$$\vec{B} \times \vec{A} = -(\vec{A} \times \vec{B}) \quad (2)$$

2. For any scalar c and vectors \vec{A} and \vec{B} ,

$$c(\vec{A} \times \vec{B}) = (c\vec{A}) \times \vec{B} = \vec{A} \times (c\vec{B}) \quad (3)$$

3. For any \vec{A} ,

$$\vec{A} \times \vec{A} = \vec{0} \quad (4)$$

$$\Rightarrow \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

$$4. \hat{i} \times \hat{j} = \hat{k} \quad \hat{j} \times \hat{i} = -\hat{k}$$

$$\hat{i} \times \hat{k} = -\hat{j} \quad \hat{i} \times (-\hat{k}) = \hat{j}$$

$$\hat{j} \times \hat{k} = \hat{i} \quad \hat{j} \times (-\hat{k}) = -\hat{i}$$

5. For arbitrary vectors in 3-D, $\vec{A} = \langle A_x, A_y, A_z \rangle$ and $\vec{B} = \langle B_x, B_y, B_z \rangle$,

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y) \hat{i} + (A_z B_x - A_x B_z) \hat{j} + (A_x B_y - A_y B_x) \hat{k} \quad (5)$$

Torque

Definition: When a force \vec{F} acts on an object at a point *on* the object that is a distance r_{\perp} away from the axis of rotation, the **torque**, τ , produced by \vec{F} is:

$$\tau \equiv Fr_{\perp} = Fr \sin \phi \quad (6)$$

Notes:

- Eq. (6) gives just the *magnitude* of the torque *vector*.
- In Eq. (6), F represents the *magnitude* of the force vector, \vec{F} . Thus, F is intrinsically positive.
- The distance r_{\perp} is called the **moment arm**, or **lever arm**, of the force \vec{F} . It is the *perpendicular* distance from the “line of action” of \vec{F} to the axis of rotation. The **line of action** of \vec{F} is simply the line along which \vec{F} acts.
- ϕ is the (smallest) angle from the *position vector* \vec{r} of the point where \vec{F} is applied to the force vector \vec{F} itself. r represents the *magnitude* of this position vector.

Sign Convention for Torque

We need to keep track of whether the torque tends to rotate the object in the clockwise (CW) sense or the counter-clockwise (CCW) sense. To do this, we assign a + or – sign to the torque:

- τ is **positive** if it tends to rotate the object in the **CCW** sense.
- τ is **negative** if it tends to rotate the object in the **CW** sense.

Net Torque

If there are N forces $\vec{F}_1, \dots, \vec{F}_N$ producing a torque on some object, then the magnitude of the **net torque** on the object is given by:

$$\tau_{net} = \sum_{i=1}^N \tau_i = \pm F_1 r_{\perp 1} \pm F_2 r_{\perp 2} \pm \dots \pm F_N r_{\perp N} \quad (7)$$

Vector Definition of Torque

Notice that Eq. (6) looks like the magnitude of a cross product. Accordingly, we can define a **vector torque**, $\vec{\tau}$, as follows:

$$\vec{\tau} \equiv \vec{r} \times \vec{F} \quad (8)$$

Note:

- \vec{r} is the *position vector* of the point at which \vec{F} is applied.

The magnitude of $\vec{\tau}$ is given by (6). The *direction* of the vector $\vec{\tau}$ is given by the RHR for the cross product:

1. Point fingers of your right hand in the direction of \vec{r} .
2. Curl fingers toward \vec{F} .
3. Thumb points in direction of $\vec{\tau}$.

It is also useful to define a vector angular velocity $\vec{\omega}$ and angular acceleration $\vec{\alpha}$.

The **vector angular velocity** $\vec{\omega}$ of an object is defined to be a vector having magnitude equal to the angular speed of the object:

$$|\vec{\omega}| \equiv \left| \frac{d\theta}{dt} \right| \quad (9)$$

The *direction* of $\vec{\omega}$ is given by a RHR:

- Curl the fingers of your right hand in the sense in which the object rotates.
- Your thumb points in the direction of $\vec{\omega}$.

The **vector angular acceleration** $\vec{\alpha}$ is defined to be the vector

$$\vec{\alpha} \equiv \frac{d\vec{\omega}}{dt} \quad (10)$$

If the object is rotating about a **fixed axis**, then the direction of $\vec{\alpha}$ is as follows:

- $\vec{\alpha}$ points in **same direction** as $\vec{\omega}$ if object is **speeding up**.
- $\vec{\alpha}$ points in **the opposite direction** if object is **slowing down**.

Angular Momentum

Angular Momentum of a Single Particle

The **angular momentum** \vec{L} of a particle having position vector \vec{r} (relative to some origin O) and moving with linear momentum \vec{p} is defined to be:

$$\vec{L} \equiv \vec{r} \times \vec{p} \quad (11)$$

From the definition of the cross product, the magnitude of \vec{L} is:

$$L = |\vec{L}| = r p \sin \phi = r m v \sin \phi, \quad (12)$$

in which ϕ is the (smallest) angle between \vec{r} and \vec{p} .

The *direction* of \vec{L} is given by the RHR for the cross product.

Unit (SI): $\text{kg} \cdot \frac{\text{m}^2}{\text{s}}$

Angular Momentum of a Rigid Body Rotating About a Fixed Axis

For rotations of a rigid body about a fixed axis, every particle in the body moves in a **circular path**.

In this case, $\phi = 90^\circ$ and (12) becomes:

$$L = rp = rmv \quad (13)$$

Consider a flat, rigid object in the x - y plane, rotating about a symmetry axis. The magnitude of the angular momentum of the i -th particle is:

$$L_i = r_i m_i v_i = (m_i r_i^2) \omega$$

The magnitude of the *total* angular momentum of the rigid body (imagined to be made up of N particles) is then:

$$L = \sum_{i=1}^N (m_i r_i^2) \omega = \omega \left[\sum_{i=1}^N (m_i r_i^2) \right]$$
$$L = I \omega \quad (14)$$

In terms of vector quantities:

$$\vec{L} = I \vec{\omega} \quad (15)$$

Eq. (15) also holds for a 3-D (i.e., non-flat) object rotating about a fixed symmetry axis, but showing that this is true is a little more difficult.

\vec{L} is an important quantity for two reasons:

1. It shows up in the “rotational form” of Newton’s 2nd law.
2. Sometimes, \vec{L} is **conserved!**

The “Rotational Form” of Newton’s 2nd Law

Consider a single particle having mass m , position vector \vec{r} (relative to some origin O), and moving with linear momentum \vec{p} .

The angular momentum of this particle with respect to the origin O is:

$$\vec{L} = \vec{r} \times \vec{p}$$

Now differentiate both sides:

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) \quad (*)$$

On the RHS, we have the derivative of a product; $\vec{r} = \vec{r}(t)$ and $\vec{p} = \vec{p}(t)$ are both functions of time, in general. Recall the **product rule** for taking the derivative of a product of real-valued functions:

$$\frac{d}{dt}[f(t)g(t)] = \left[\frac{d}{dt}f(t)\right]g(t) + f(t)\left[\frac{d}{dt}g(t)\right]$$

In a similar way, (*) becomes:

$$\frac{d\vec{L}}{dt} = \left(\frac{d\vec{r}}{dt} \times \vec{p}\right) + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right)$$

$$\frac{d\vec{L}}{dt} = (\vec{v} \times m\vec{v}) + (\vec{r} \times \vec{F}_{net})$$

But $\vec{v} \times m\vec{v} = \vec{0}$, so we get:

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}_{net}, \quad (**)$$

in which $\vec{F}_{net} = d\vec{p}/dt$ is the net force on this single particle.

The quantity on the RHS of (**) is the net torque $\vec{\tau}_{net}$ on this particle. So we finally have:

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt} \quad (16)$$

This is the “rotational form” of Newton’s 2nd law, $\vec{F}_{net} = d\vec{p}/dt$. It was derived here for a single particle, but the same result holds (summing over all particles) for any *system* of particles (such as a rigid body), except that $\vec{\tau}_{net}$ becomes just the net *external* torque. So for the whole system of particles:

$$(\vec{\tau}_{net})_{ext} = \frac{d\vec{L}}{dt} \quad (17)$$

Conservation of Angular Momentum

Angular momentum is an important physical quantity because sometimes it is conserved. But when (i.e., under what condition(s))? The answer follows at once from the rotational form of Newton's 2nd law, (17). If \vec{L} is conserved, then

$$\frac{d\vec{L}}{dt} = \vec{0}$$

and, from (17):

$$(\vec{\tau}_{net})_{ext} = \vec{0} \quad (18)$$

The angular momentum of an object (or *system*) is **conserved** whenever the **net external torque on the object (or system) is zero**. This observation is called the **law of conservation of angular momentum**. It is one of the most important statements in the entire course.

For a rigid body rotating about a fixed symmetry axis, if \vec{L} is conserved:

$$\begin{aligned} L_i &= L_f \\ I_i \omega_i &= I_f \omega_f \end{aligned} \quad (19)$$

The Rotational Form of Newton's 2nd Law (revisited)

We found earlier that for any system of particles:

$$(\vec{\tau}_{net})_{ext} = \frac{d\vec{L}}{dt} \quad (20)$$

This is in every way analogous to what I will now call the “translational form” of Newton's 2nd law:

$$(\vec{F}_{net})_{ext} = \frac{d\vec{P}}{dt}$$

For a rigid body rotating about a fixed symmetry axis, we also found that

$$L = I\omega$$

Taking the magnitude of both sides of (20), then, I find:

$$(\tau_{net})_{ext} = \frac{d}{dt}(I\omega)$$

For a rigid body rotating about a fixed axis, I is constant, so:

$$(\tau_{net})_{ext} = I \frac{d\omega}{dt} = I\alpha$$

For a rigid body, every torque you apply to it is an *external* torque. So:

$$\tau_{net} = I\alpha \quad (21)$$

This is completely analogous to $F_{net} = ma$.

Torque Due to the Weight of a Rigid Body

In problems to be considered soon, we will need to be able to calculate the torque on a rigid body caused by the body's own weight.

Treating the rigid body as a system of N particles, the torque on the entire rigid body due to the gravitational force would be written:

$$\vec{\tau}_{grav} = \sum_{i=1}^N \vec{r}_i \times (m_i \vec{g}),$$

in which \vec{g} is a vector of magnitude 9.81 m/s^2 , downward.

Rewriting a little, I get:

$$\begin{aligned} \vec{\tau}_{grav} &= \left[\sum_{i=1}^N (m_i \vec{r}_i) \right] \times \vec{g} = M_{tot} \vec{r}_{CM} \times \vec{g} \\ \vec{\tau}_{grav} &= \vec{r}_{CM} \times M_{tot} \vec{g}, \end{aligned} \tag{22}$$

which says that the **torque is the same as if all the mass were concentrated at the CM!**

Note slight subtlety here... I assumed that \vec{g} was the same at the locations of all the particles. If this is not true, then we must be more careful... The **center of gravity** is not the same as the **center of mass**.

Combined Rotation and Translation

In some situations (rolling without slipping, e.g.), the motion of a system is partly translational and partly rotational. In such situations, it is always possible to split up the total kinetic energy of the system into two parts: the **translational** kinetic energy **of the center of mass** and the **rotational** kinetic energy **about the center of mass**:

$$\begin{aligned} K_{tot} &= K_{trans} + K_{rot} \\ K_{tot} &= \frac{1}{2} M v_{CM}^2 + \frac{1}{2} I_{CM} \omega^2 \end{aligned} \quad (23)$$

Rolling Without Slipping

Rolling without slipping is a combination of a pure translation and a pure rotation. If the wheel is not to slip (skid), then at every instant, the **point of contact** between the wheel and the surface on which it rolls must be **momentarily at rest**. At every instant, it is as though the wheel were rotating about an axis passing through the point of contact.

One consequence of the above discussion is that the translational speed of the center of mass, v_{CM} , is related to the angular speed ω of the entire wheel by:

$$v_{CM} = R\omega, \quad (24)$$

in which R is the radius of the wheel.

Eq. (24) holds at every instant, as long as the wheel rolls without slipping. Therefore, it must also be true at every instant that:

$$\begin{aligned} \frac{dv_{CM}}{dt} &= R \frac{d\omega}{dt} \\ a_{CM} &= R\alpha \end{aligned} \quad (25)$$