Physics 1401 Lecture Notes – Math Review

In the following notes, I hope to refresh your memory concerning a few mathematical issues that we will have to deal with in this course. I also hope to bring to light and help you avoid some common mistakes that students in Physics 1401 make frequently. I've organized the topics by subject area: first algebra, then geometry, and finally trigonometry.

Algebra

1. **Common mistake: Improper cancellation with fractions.** Given an expression like:

   $$\frac{a}{a+b},$$

   you **cannot** cancel the a's:

   $$\frac{a}{a+b} \neq \frac{\not{a}}{\not{a}+b}$$

   so:

   $$\frac{a}{a+b} \neq \frac{1}{b}$$

   For example,

   $$\frac{2}{2+4} \neq \frac{1}{4}.$$

   The *correct* way to reduce this fraction is as follows:

   $$\frac{2}{2+4} = \frac{2}{6} = \frac{1}{3}.$$

2. When given an expression like:

   $$\frac{a+b}{c}$$

   you can rewrite this as two separate fractions:

   $$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}.$$

   For example,

   $$\frac{15+10}{5} = \frac{15}{5} + \frac{10}{5}$$

   $$\frac{15+10}{5} = 3+2$$

   $$\frac{15+10}{5} = 5,$$

   as you can readily verify by adding the numbers in the numerator first and dividing the result by 5.

3. **Getting common denominators.** Suppose you want to perform the following sum:

   $$\frac{a}{b} + \frac{c}{d} = ?$$

   In order to add these two fractions, you must first get a **common denominator**. (That is, you must rewrite the two fractions so that they have the *same* denominator). One way to do this which *always* works is to let the common denominator be the product of the two individual denominators. (This would be the product *bd* in the example above.) Rewriting each fraction with this denominator gives:

   $$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{bd}$$

So the answer is:

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+cb}{bd}.$$

If you'd like a *numerical* example, consider (just to make up some numbers):

$$\frac{1}{2}+\frac{1}{3}$$

Getting a common denominator, I find:

$$\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6},$$

so:

$$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}.$$

4. **Getting common denominators (revisited).** When adding whole numbers and fractions, you must first get a common denominator. For example, if you want to perform the sum:

$$1+\frac{3}{4},$$

first rewrite the 1 as follows:

$$\frac{4}{4}+\frac{3}{4}.$$

Now add the fractions:

$$\frac{4}{4}+\frac{3}{4}=\frac{7}{4}.$$

5. **Factoring.** Suppose you want to solve the following equation for x:

$$3(x+1)=15-2(x+1)$$

One way to do this would be to gather the terms involving x on one side of the equation:

$$3(x+1)+2(x+1)=15$$

Then notice that the two terms on the left-hand side have a common factor of $(x+1)$. Factoring out $(x+1)$ on the left-hand side, we can rewrite the equation:

$$(x+1)(3+2)=15,$$

or:

$$(x+1)(5)=15.$$

Then solving for $x+1$ gives:

$$x+1=3,$$

so:

$$x=2.$$

6. **Solving pairs of simultaneous equations.** Solve the following pair of simultaneous equations. (That is, find the values of x and y which satisfy *both* equations.)

$$x+2y=8 \qquad (1)$$
$$x+y=6 \qquad (2)$$

There are several ways of solving this (or any other) pair of simultaneous equations, but probably the most useful for us is the following:

- solve one of the equations for one of the unknowns. For example, solve Eq. (1) above for x.

- then substitute the resulting expression into the other of the two equations. In doing so, you'll *eliminate* one of the unknowns from the second equation, leaving only one unknown. For example, if we solve Eq. (1) for x and substitute this expression for x into Eq. (2), x will be eliminated from Eq. (2), leaving y as the only remaining unknown in Eq. (2).

For example, solving Eq. (1) for x gives:

$$x = 8 - 2y . \qquad (3)$$

Substituting Eq. (3) into Eq. (2) gives:

$$8 - 2y + y = 6 .$$

(Notice that the unknown x no longer appears in this equation.) Now just solve this equation for y:

$$8 - 6 = y$$

$$\boxed{y = 2}.$$

To find *x*, we could substitute this value of y back into Eq. (1) or Eq. (2). (After all, it's supposed to satisfy *both* of these equations.) But probably the *easiest* thing to do is to substitute it back into Eq. (3), since this already gives *x* in terms of *y*:

$$x = 8 - 2(2)$$

$$\boxed{x = 4}.$$

7. **Solving quadratic equations (Part I: Factoring).** First of all, a definition: a *quadratic* equation is a polynomial equation of order 2. This means that the highest power of the unknown appearing in the equation is the *second* power. For example, $2x^2 = -3x - 1$ is a quadratic equation. In general, any quadratic equation has *two* solutions. The solutions may have different values, or they may turn out to be numerically equal to one another. The solutions may be real numbers, or they may turn out to be imaginary numbers. In this course, the only quadratic equations we will *ever* encounter will be ones with *real* solutions.

It should be apparent (perhaps after a few moments' thought) that *any* quadratic equation in the unknown *x* can be written in the form:

$$ax^2 + bx + c = 0 , \qquad (4)$$

for some coefficients *a*, *b*, and *c*. For example, the quadratic equation given above can be rewritten as:

$$2x^2 + 3x + 1 = 0 , \qquad (5)$$

which is in the form of Eq. (4), with $a = 2$, $b = 3$, and $c = 1$.

Factoring a quadratic equation means to write it in the form of Eq. (4), then rewrite the left-hand side as the product of two *binomials*. For example, Eq. (5) can be rewritten as:

$$(2x+1)(x+1) = 0 , \qquad (6)$$

as you can verify by multiplying the two binomials $(2x+1)$ and $(x+1)$ to see that their product is, in fact, equal to $2x^2 + 3x + 1$.

Finally, having written our quadratic equation in the form of Eq. (6), we can easily find the solutions (the values of *x* that satisfy it). After all, $2x + 1$ and $x + 1$ are just some *numbers* (for a given value of *x*), and Eq. (6) says that the product of these two numbers must equal zero. Well, the only way for this to be true is if one or the other number is *itself* equal to zero! That is, in order for Eq. (6) to be true, it *must* be true either that:

$$2x + 1 = 0 \qquad (7)$$

or that:

$$x + 1 = 0 . \qquad (8)$$

Solving Eq. (7) for *x* gives one solution:

$$\boxed{x=-\frac{1}{2}}.$$

Eq. (8) gives the other solution:

$$\boxed{x=-1}.$$

8. **Solving quadratic equations (Part II: using the *quadratic formula*).** Sometimes, a quadratic equation cannot be factored into two binomials. For example, one of the equations describing the position y of an object in free fall (at any time t) says:

$$y=y_0+v_0t-\frac{1}{2}gt^2, \qquad (9)$$

in which y_0 is the initial position, v_0 is the initial velocity, and g is the acceleration due to gravity. (Note that Eq. (9) is a quadratic equation in the variable t.) Suppose you throw a baseball upward from an initial position we will call $y_0=0$ with an initial velocity $v_0=15\text{ m/s}$. At what time(s) will the ball be 1.0 m above its release point? Well, if at some time t, the ball's position is $y=1.0\text{ m}$, then Eq. (9) says:

$$1.0\text{ m}=0+(15\text{ m/s})t-\frac{1}{2}\left(9.81\text{ m/s}^2\right)t^2,$$

which can be rewritten in the form $at^2+bt+c=0$ as follows:

$$\left(4.91\text{ m/s}^2\right)t^2-(15\text{ m/s})t+1.0\text{ m}=0. \qquad (10)$$

If you look at Eq. (10) for even a few seconds, you realize that it cannot be factored into the product of two binomials.

In this situation, we can solve Eq. (10) using the *quadratic formula*, which says that for any quadratic equation of the form $ax^2+bx+c=0$, the two solutions are:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

(One solution is $x=\frac{-b+\sqrt{b^2-4ac}}{2a}$; the other is $x=\frac{-b-\sqrt{b^2-4ac}}{2a}$.) So for Eq. (10), the solutions would be:

$$t=\frac{(15\text{ m/s})\pm\sqrt{(-15\text{ m/s})^2-4\left(4.91\text{ m/s}^2\right)(1.0\text{ m})}}{2\left(4.91\text{ m/s}^2\right)}$$

Doing the math gives the two solutions as:

$$\boxed{t=3.0\text{ s}}$$

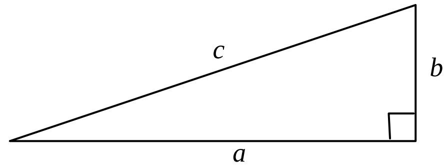
and:

$$\boxed{t=0.068\text{ s}}$$

The second of these solutions is the time when the ball passes through $y=1.0\text{ m}$ *going up*. The first solution is the time when it passes through $y=1.0\text{ m}$ *going down*.

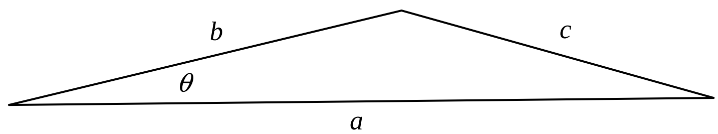
Geometry

1. **Pythagorean theorem.** Consider a right triangle with hypotenuse of length *c*, the other two sides of which have lengths *a* and *b*:



The Pythagorean theorem says $\boxed{c^2 = a^2 + b^2}$. (Note that this applies *only* to *right* triangles.)

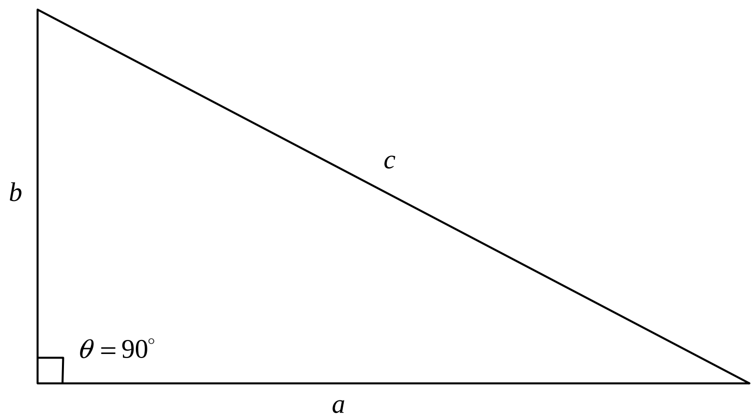
2. **Law of cosines.** (Like the Pythagorean theorem, but applies to *any* triangle, whether it's a right triangle or not.) Consider the triangle below:



The Law of Cosines says:

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos\theta}.$$

Notice that this is similar to the Pythagorean theorem, but it includes the additional term $-2ab\cos\theta$. Also note that there's an implied connection between the side that you call "*c*" and the angle that you call "θ": the side called "*c*" is understood to be the side *opposite* the angle called "θ". Finally, note that the law of cosines *includes* the Pythagorean theorem as a *special case*. For if you imagine "opening up" the angle θ until it becomes 90°, the triangle would become a *right* triangle:



and the law of cosines would become:

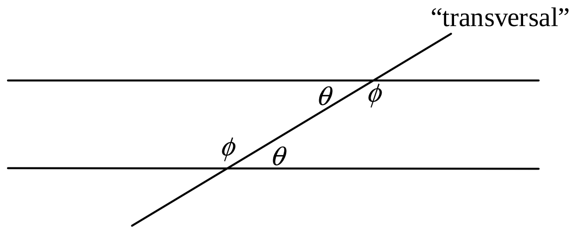
$$c^2 = a^2 + b^2 - 2ab\cos 90^\circ.$$

But, since $\cos 90^\circ = 0$, this is equivalent to:

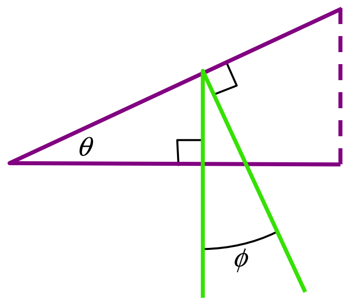
$$c^2 = a^2 + b^2,$$

which is the Pythagorean theorem.

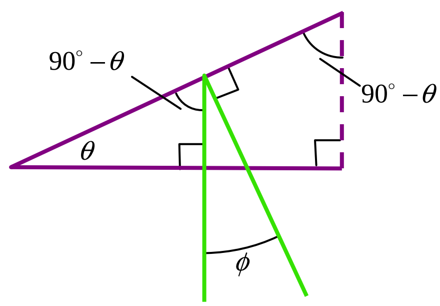
3. **If parallel lines are cut by a transversal, alternate interior angles are equal.** (The *transversal* is just a line that *cuts across* the two parallel lines.)



4. Consider the pair of solid purple lines shown below, the angle between which is θ. Now consider a second set of lines, shown in green, each of which is perpendicular to one of the lines in the first pair, as shown. What is the angle between the second pair of lines? (That is, what is the angle ϕ?)



Well, consider making a right triangle out of the pair of purple lines by adding the *dashed* purple line shown in the picture above. Clearly, then, the angle between the upward-sloping purple line and the dashed line is $90^\circ - \theta$. This angle is the angle between the upward-sloping purple line and the vertical. But one of the *green* lines is *also* vertical, so the angle between *it* and the upward-sloping purple line must *also* be $90^\circ - \theta$, as shown below.



But *this* angle plus ϕ must equal 90°, from the picture, so we must have:

$$\left(90^\circ - \theta\right) + \phi = 90^\circ ,$$

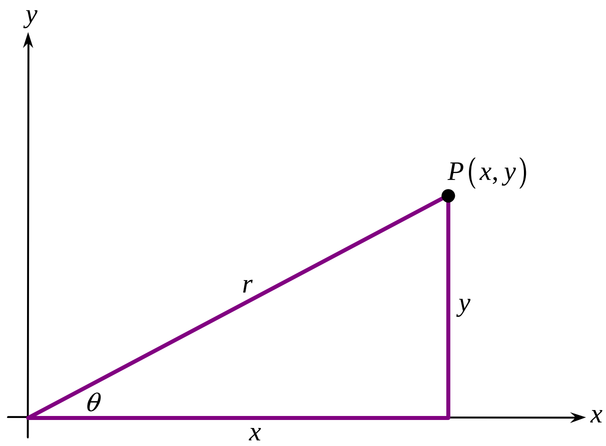
which, when solved for ϕ, gives:

$$\boxed{\phi = \theta}.$$

So, in fact, the angle between the two green lines is *the same as* the angle between the two purple lines.

Trigonometry

1. **Definitions of sine, cosine, and tangent of an angle.** Consider the right triangle shown drawn on an *x*-*y* coordinate system below. If the point *P* has coordinates (x, y), as shown, then the horizontal side of the right triangle has length *x* and the vertical side has length *y*, as shown. The hypotenuse is called *r* in the figure below.



The **sine, cosine, and tangent** of the angle θ are defined as follows:

- *sine of* θ: ratio of length of *opposite* side to length of *hypotenuse*:

$$\boxed{\sin\theta \equiv \frac{y}{r}}$$

- *cosine of* θ: ratio of length of *adjacent* side to length of *hypotenuse*:

$$\boxed{\cos\theta \equiv \frac{x}{r}}$$

- *tangent of* θ: ratio of length of *opposite* side to length of *adjacent* side:

$$\boxed{\tan\theta \equiv \frac{y}{x}}$$

Note that since these are all defined as ratios of the lengths of sides of the triangle, $\sin\theta$, $\cos\theta$, and $\tan\theta$ are all *dimensionless* (unitless)! They're just *numbers*!

2. Suppose you know that:

$$\sin\theta = \frac{1}{2}, \qquad (11)$$

for some angle θ. How do you find out what the angle θ is? Answer: take the *inverse sine* of both sides of Eq. (11). The *inverse sine*, written "$\sin^{-1}$" (or sometimes, "arcsin") is defined by the property that if you take the inverse sine of the sine of an angle, you get the angle back again:

$$\sin^{-1}(\sin\theta) = \theta$$

So if we take the inverse sine of both sides of Eq. (11), we get:

$$\sin^{-1}(\sin\theta) = \sin^{-1}\left(\frac{1}{2}\right)$$

$$\boxed{\theta = \sin^{-1}\left(\frac{1}{2}\right) = 30^\circ}$$

Similarly, if you know that $\cos\theta = \frac{1}{2}$, you can find θ by taking the *inverse cosine*:

$$\cos^{-1}(\cos\theta) = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\boxed{\theta = 60^\circ}$$

And if you have $\tan\theta = \frac{1}{2}$, you can find θ by taking the *inverse tangent*:

$$\tan^{-1}(\tan\theta) = \tan^{-1}\left(\frac{1}{2}\right)$$

$$\boxed{\theta = \tan^{-1}\left(\frac{1}{2}\right) \approx 26.6^\circ}$$