Handout for Rob Eby's talk

Group Exams but an Individual Final Exam: How Does That Work? S 170 - 2:15 3:05

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Some problems adapted from *Calculus Problems for a New Century* published by the MAA and *ConcepTests* by Hughes-Hallett, Gleason, McCallum, et al.

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Example Take Home Questions For Liberal Arts Mathematics

Take home 1 - LA A yardstick measures $\frac{1}{4}$ by 3 by 36 inches. How many yard sticks will fit in a box:

(a) 3 inches wide and 36 inches high, if the girth of the box is 30 inches?

(b) 6 inches wide and 36 inches high, if the girth of the box is 30 inches?

(c) 3 inches wide and 72 inches high, if the girth of the box is 30 inches?

(d) 3 inches wide and 36 inches high, if the girth of the box is 60 inches?

(e) 6 inches wide and 72 inches high, if the girth of the box is 30 inches?

(f) 6 inches wide and 36 inches high, if the girth of the box is 60 inches?

(g) 3 inches wide and 72 inches high, if the girth of the box is 60 inches?

Take home 2 - LA Rob and Lydia are going to play a game. They roll one fair die. If an even number comes up, Lydia gets \$1 from Rob. If an odd number comes up, Rob gets \$1 from Lydia. Find the expected value of this game from Rob's perspective in each of the following cases:

(a) The die has six sides

(b) The die has seven sides

(c) The die has eight sides

(d) The die has nine sides

(e) The die has ten sides

Take home 3 - LA Rob and Lydia are going to play a new game. They roll one fair die. If an even number comes up, Lydia gets the amount showing in dollars. (So \$4 if a four comes up) If an odd number comes up, Rob gets the amount showing in dollars. (So \$3 if a three comes up) Find the expected value of this game from Rob's perspective in each of the following cases:

(a) The die has six sides

(b) The die has seven sides

(c) The die has eight sides

(d) The die has nine sides

(e) The die has ten sides

In Class Exam for Liberal Arts Mathematics

Problem 1 (Liberal Arts Mathematics) On the take home part you were given the following problem:

A yardstick measures $\frac{1}{4}$ by 3 by 36 inches. How many yard sticks will fit in a box:

(a) 3 inches wide and 36 inches high, if the girth of the box is 30 inches? _______ 120

(b) 6 inches wide and 36 inches high, if the girth of the box is 30 inches? _______ 240

(c) 3 inches wide and 72 inches high, if the girth of the box is 30 inches? _______ 240

(d) 3 inches wide and 36 inches high, if the girth of the box is 60 inches? _______ 240

(e) 6 inches wide and 72 inches high, if the girth of the box is 30 inches? _______ 480

(f) 6 inches wide and 36 inches high, if the girth of the box is 60 inches? _______ 480

(g) 3 inches wide and 72 inches high, if the girth of the box is 60 inches? _______ 480

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(1) So how many of these yardsticks will fit into a box that is 6 inches wide and 72 inches high, if the girth of the box is 60 inches?
(2) How many will fit into a box that is 3*R inches wide and 36*R inches high, if the girth of the box is 30*R inches, where R is some positive number greater than one? Explain how you know!

Problem 2 (Liberal Arts Mathematics) On the take home part you were given the following problem:

Rob and Lydia are going to play a game. They roll one fair die. If an even number comes up, Lydia gets $1 from Rob. If an odd number comes up, Rob gets $1 from Lydia. Find the expected value of this game from Rob's perspective in each of the following cases:

(a) The die has six sides _______ $(\frac{3}{6}(1) + \frac{3}{6}(-1) = 0$

(b) The die has seven sides _______ $(\frac{3}{7}(-1) + \frac{4}{7}(1) = \frac{1}{7} = 0.143$

(c) The die has eight sides _______ $(\frac{4}{8}(1) + \frac{4}{8}(-1) = 0$

(d) The die has nine sides _______ $(\frac{4}{9}(-1) + \frac{5}{9}(1) = \frac{1}{9} = 0.111$

(e) The die has ten sides _______ $(\frac{5}{10}(1) + \frac{5}{10}(-1) = 0$

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(1) Explain, without actually computing it, why the expected value for ANY die with an even number of sides will end up with an expected value of zero but a die with an odd number of sides will always be in Rob's favor.

Problem 3 (Liberal Arts Mathematics) Rob and Lydia are going to play a new game. They roll one fair die. If an even number comes up, Lydia gets the amount showing in dollars. If an odd number comes up, Rob gets the amount showing in dollars. Find the expected value of this game from Rob's perspective in each of the following cases:

(a) The die has six sides _______ $\frac{1}{6}(1) + \frac{1}{6}(-2) + \frac{1}{6}(3) + \frac{1}{6}(-4) + \frac{1}{6}(5) + \frac{1}{6}(-6) = -\frac{1}{2} = -0.50$

(b) The die has seven sides _______ $\frac{1}{7}(1) + \frac{1}{7}(-2) + \frac{1}{7}(3) + \frac{1}{7}(-4) + \frac{1}{7}(5) + \frac{1}{7}(-6) + \frac{1}{7}(7) = \frac{4}{7} = 0.571$

(c) The die has eight sides _______ $\frac{1}{8}(1) + \frac{1}{8}(-2) + \frac{1}{8}(3) + \frac{1}{8}(-4) + \frac{1}{8}(5) + \frac{1}{8}(-6) + \frac{1}{8}(7) + \frac{1}{8}(-8) = -\frac{1}{2} = -0.50$

(d) The die has nine sides _______ $\frac{1}{9}(1) + \frac{1}{9}(-2) + \frac{1}{9}(3) + \frac{1}{9}(-4) + \frac{1}{9}(5) + \frac{1}{9}(-6) + \frac{1}{9}(7) + \frac{1}{9}(-8) + \frac{1}{9}(9) = \frac{5}{9} = 0.556$

(e) The die has ten sides _______ $\frac{1}{10}(1) + \frac{1}{10}(-2) + \frac{1}{10}(3) + \frac{1}{10}(-4) + \frac{1}{10}(5) + \frac{1}{10}(-6) + \frac{1}{10}(7) + \frac{1}{10}(-8) + \frac{1}{10}(9) + \frac{1}{10}(-10) = -\frac{1}{2} = -0.50$

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(1) Explain why for ANY die with an even number of sides Rob is in a losing game but for ANY die with an odd number of sides Rob is in a winning game.

Other Uses For the Take Home Part of the Exam

Problem 4 (Liberal Arts Mathematics OR Statistics) On the take home part you were given the following problem:

Each of the following are for a data set that is normally distributed. In each case, compute the area between r_1 and r_2. Make note of any major patterns you see.
Set One

- I $r_1 = 80$, $r_2 = 120$, mean=100, standard deviation=5 : NCDF(80,120,100,5)=0.999937
- II $r_1 = 80$, $r_2 = 120$, mean=100, standard deviation=10 : NCDF(80,120,100,10)=0.9545
- III $r_1 = 80$, $r_2 = 120$, mean=100, standard deviation=20 : NCDF(80,120,100,20)=0.6827
- IV $r_1 = 80$, $r_2 = 120$, mean=100, standard deviation=40 : NCDF(80,120,100,40)=0.3829

Set Two

- A $r_1 = 60$, $r_2 = 140$, mean=100, standard deviation=10 : NCDF(60,140,100,5)=0.999937
- B $r_1 = 80$, $r_2 = 120$, mean=100, standard deviation=10 : NCDF(80,120,100,5)=0.9545
- C $r_1 = 90$, $r_2 = 110$, mean=100, standard deviation=10 : NCDF(90,110,100,5)=0.6827
- D $r_1 = 95$, $r_2 = 105$, mean=100, standard deviation=10 : NCDF(95,105,100,5)=0.3829

In I there was a distance of 40 between r_1 and r_2 while in A there was a distance of 80 between r_1 and r_2. Yet the probability is the same. In fact, if we continue for the pairs (II-B, III-C, IV-D) we find that the probability is the same yet the distances are different. Explain why this is NOT surprising in terms of what we discussed in class about normal distributions.

Problem 5 (Statistics) Estimating a Population Size. The Story: (NPR, April 11, 2009) California's salmon season has been officially called off for the second straight year. Fishery managers canceled all commercial and sport fishing as scientists struggle to figure out why the Chinook salmon population has collapsed. The decision to shut down the season was a relatively easy one, says Frank Lockhart, head of sustainable fisheries for the Pacific Fisheries Management Council. The Chinook salmon population has declined dramatically over the past two years. The Question: How do they know the current Chinook salmon population? One Method: Tag, Release, then sample.

- I Start by tagging a known number of salmon. Call this number y.
- II Let N denote the salmon population in the given region (unknown).
- III Then the population proportion of tagged fish is $p = y/N$.
- IV Later, sample the population of salmon. You let the fisherman do this. They are required, by law, to report how many salmon they have caught, and how many of these were tagged.
- V Let n (sample size) be the total number of salmon caught, and x (successes) denote the number of these that are tagged. The proportion of tagged fish in the sample is then $\widehat{p} = x/n$.
- VI It stands to reason that $\widehat{p} \approx p = y/N$.
- VII Now, you know $\widehat{p}$ and y, then you rearrange the equation to be $N \approx y/\widehat{p}$. There you have a point estimate for the number of salmon.
- VIII To improve matters, you calculate a 95% confidence interval for p. Use the upper limit and lower limit to get a 95% confidence interval for the number of salmon. 4. Give it a try: Suppose the Pacific Fisheries Management Council tags 5,000 salmon before the start of the season. During the season, fisherman report that of 1,562,500 salmon caught, 1,250 were tagged. Find the point estimate for N (the population size) and the 95% confidence interval for N.

First we start by finding the point estimate for the population. So in this case $y = 5000$. Then based upon the results of the fishermen $\widehat{p} = 1250/1562500 = 0.0008$ or 8e-4. This means $N = y/\widehat{p} = 5000/0.0008 = 6250000$.
Now to find a 95% confidence interval for the population, we start by finding a 95% confidence interval for the true population proportion, then use the above procedure twice.
1 proportion z interval with $x = 1250, n = 1562500$ gives $(0.00076, 0.00084)$ or (7.6e-4, 8.4e-4). Now to find the 95% confidence interval for the actual population we work as above to get (5,952,380.952, 6,578,947.368)

Statistics Take Home Questions

Problem 6 Statistics For each of the following set ups, run the hypothesis test and report the p-value. (So I only need a p-value and conclusion as your answer for each one)

1. A die is rolled 100 times. The total of the spots is 368 instead of the expected 350. Is the die loaded?
2. A die is rolled 200 times. The total of the spots is 736 instead of the expected 700. Is the die loaded?
3. A die is rolled 500 times. The total of the spots is 1840 instead of the expected 1750. Is the die loaded?
4. A die is rolled 1000 times. The total of the spots is 3680 instead of the expected 3500. Is the die loaded?

Problem 7 Statistics A box contains 1 maroon marble and 99 white marbles. 100 marbles are drawn **with replacement**.

1. Find the expected number of maroon marbles in the 100 draws and the standard error
2. What is the chance of drawing fewer than 0 maroon marbles?

Now we get a new box, and are told there are 10,000 marbles in it, some of which are maroon and some of which are white. However we cannot see into the box, only reach in and grab a marble. (Yes, it is a strange box, deal with it!) To estimate the percentage of maroon marbles, 100 are drawn without replacement. Only 1 turns out to be maroon.

1. Is this binomial? Explain.
2. What did we discuss in class about these kind of situations and the binomial distribution?
3. Based upon the data, find a 95% confidence interval for the true percentage of maroon marbles in the box.

In Class Exam Statistics

Problem 8 Statistics On the take home part of the test you were given the following problem:

For each of the following set ups, run the hypothesis test and report the p-value. (So I only need a p-value and conclusion as your answer for each one)

1. A die is rolled 100 times. The total of the spots is 368 instead of the expected 350. Is the die loaded?
   Z-test 3.5, 1.7078, 368/100=3.68, 100, $\neq$ gives p-value = 0.2919 Fail to Reject
2. A die is rolled 200 times. The total of the spots is 736 instead of the expected 700. Is the die loaded?
   Z-test 3.5, 1.7078, 736/200=3.68, 100, $\neq$ gives p-value = 0.1361 Fail to Reject
3. A die is rolled 500 times. The total of the spots is 1840 instead of the expected 1750. Is the die loaded?
   Z-test 3.5, 1.7078, 1840/500=3.68, 100, $\neq$ gives p-value = 0.0184 Reject
4. A die is rolled 1000 times. The total of the spots is 3680 instead of the expected 3500. Is the die loaded?
   Z-test 3.5, 1.7078, 3680/1000=3.68, 100, $\neq$ gives p-value = 0.000859 Reject

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Given the above information: (1) In each case, the sample mean $\overline{x} = 3.68$ and $H_0 : \mu = 3.5$. So explain why the p-value is decreasing.

A Consider 1 to 2. We doubled the sample size, but the p-value did NOT get cut in half but ended up less than that.

B Similar story for 1 to 3. The sample size was increased by a factor of five, but $0.2919/5 = 0.05838$ which is not the p-value.

C Likewise for 1 to 4. The sample size was increased by a factor of ten, but $0.2919/10 = 0.02919$ which again is not the p-value.

(2) So what is happening here (in A, B, and C) to cause the p-values to plummet like they are instead of going in a nice orderly way?

Problem 9 Statistics On the take home part of the exam you were given the following problem:

A box contains 1 maroon marble and 99 white marbles. 100 marbles are drawn **with replacement**.

1. Find the expected number of maroon marbles in the 100 draws and the standard error ______ This is binomial, since with replacement means the probability stays the same. So we use the formulas $\mu = np, \sigma = \sqrt{p(1-p)/n}$. This gives $\mu = 100 * 1/100 = 1, \sigma = \sqrt{\frac{1}{100}(1 - \frac{1}{100})/100} = 0.0099498744$ I used a lot of decimal places, you did not have to use that many.
2. What is the chance of drawing fewer than 0 maroon marbles? ______ None, as that would be impossible. (This should lead into number 3 of the next part)

Now we get a new box, and are told there are 10,000 marbles in it, some of which are maroon and some of which are white. However we cannot see into the box, only reach in and grab a marble. (Yes, it is a strange box, deal with it!) To estimate the percentage of maroon marbles, 100 are drawn without replacement. Only 1 turns out to be maroon.

1. Is this binomial? Explain. ______ Not quite, as the probability changes (slightly) with each new draw.
2. What did we discuss in class about these kind of situations and the binomial distribution? ______ Since we are only drawing 100 and there are 10,000 then we are not drawing that many compared to the total, (around 5% to 10% if you recall) then we can approximate the actual distribution really well with the binomial.
3. Based upon the data, find a 95% confidence interval for the true percentage of maroon marbles in the box. ______ 1 proportion z interval, $x = 1, n = 100, 95\%$ gives $(-0.0095, 0.0295)$

Clearly the probability of drawing less than one maroon marble is zero, as that is not possible. Explain (1) why then the confidence interval generated contains a negative number and (2) explain how you would 'fix' this problem if you had to use the standard procedure to find an interval that contained 'reasonable' numbers in it.

Problem 10 Calculus I Let $M = C(r)$ be the total cost of paying off a 6 year car loan that has an annual interest rate of $r\%$

I What are the units of $C'(r)$?

II What is the sign of $C(5)$?

III What is the sign of $C(7)$?

IV What is the practical meaning of $C(5)$?

V What is the practical meaning of $C(7)$?

VI Explain which of $C(5)$ and $C(7)$ is larger. If it is impossible to determine, explain why.

VII What is the sign of $C'(5)$? Explain.

VIII What is the sign of $C'(7)$? Explain.

IX Explain which of $C'(5)$ and $C'(7)$ is larger. If it is impossible to determine, explain why.

X What is the practical meaning of $(C^{-1})'(9000)$?

What is the practical meaning of $C'(5)$? Pick the best choice below.

a The rate of change of the total cost of the car is $C'(5)$.

b If the interest rate increases by 1% then the total cost of the loan increases by $C'(5)$.

c If the interest rate increases by 1% then the total cost of the loan increases by $C'(5)$ when the interest rate is 5%.

d If the interest rate increases by 5% then the total cost of the loan increases by $C'(5)$

Problem 11 (Calculus I) For each of the following functions, find the critical point(s) and inflection points(s), and label them as max, mins, etc. Make sure to mention any patterns or similarities between the three groups of functions.

I $f(x) = 2x^3 - 9x^2 - 24x + 1$

II $f(x) = 4x^3 - 15x^2 - 42x - 1$

III $f(x) = x^3 - 6x^2 + 9x - 1$

i $f(x) = 2x^3 - 9x^2 + 24x + 1$

ii $f(x) = 2x^3 - 15^2 - 42x - 1$

iii $f(x) = 2x^3 - 7x^2 + 16x + 1$

1 $f(x) = x^3 - 9x^2 + 27x + 1$

2 $f(x) = x^3 - 6x^2 + 12x - 2$

3 $f(x) = x^3 + 12x^2 + 48x - 1$

Problem 12 (Calculus I) On the take home part of the exam you had the following question: (answers provided)

Let $M = C(r)$ be the total cost of paying off a 6 year car loan that has an annual interest rate of $r\%$

I What are the units of $C'(r)$? dollars per percentage

II What is the sign of $C(5)$? positive (it costs money)

III What is the sign of $C(7)$? positive (it costs money)

IV What is the practical meaning of $C(5)$? The total cost of this six year loan at a 5% interest rate.

V What is the practical meaning of $C(7)$? The total cost of this six year loan at a 7% interest rate.

VI Explain which of $C(5)$ and $C(7)$ is larger. If it is impossible to determine, explain why. $C(7)$ is larger, as it costs more as the interest rate increases.

VII What is the sign of $C'(5)$? Explain. Positive, as the cost function is increasing.

VIII What is the sign of $C'(7)$? Explain. Positive, as the cost function is increasing.

IX Explain which of $C'(5)$ and $C'(7)$ is larger. If it is impossible to determine, explain why. $C'(7)$ is larger, as the total cost of a loan is an exponential function.

What is the practical meaning of $C'(5)$? Pick the best choice below. ***** If the interest rate increases by 1% then the total cost of the loan increases by $C'(5)$ when the interest rate is 5%. *****
What is the practical meaning of $(C^{-1})'(9000)$? First, what is $C^{-1}(9000)$? It means we find the interest rate that gives a total cost of $9000. So $(C^{-1})'(9000)$ then is how fast the interest rate is changing when the total cost of the loan is $9000.

Now let $A = f(t)$ be the depth of tread, in centimeters, on a radial tire as a function of the time elasped t, in months, since the purchase of the tire.
(I) Interpret each of the following in practical terms, paying close attention to units.
(A) $f(6) = 0.5$

(B) $f^{-1}(0.31) = 15$

(C) $f'(12) = -0.015$

(D) $(f^{-1})'(0.4) = -60$

(II) What is the sign of $f'(t)$? Explain why that is the case.

(III) What would it mean if $F'(t)$ had the opposite sign to what you answered in II above? Explain.

(IV) What is the sign of $(f^{-1})'(A)$? Explain why both in words and graphically.

Problem 13 (Calculus I) On the take home part of the exam you had the following question: (answers provided)

For each of the following functions, find the critical point(s) and inflection points(s), and label them as max, mins, etc. Make sure to mention any patterns or similarities between the three groups of functions.

I $f(x) = 2x^3 - 9x^2 - 24x + 1$ _______ critical points at $x = -1, 4$, IP at $x = 3/2$

II $f(x) = 4x^3 - 15x^2 - 42x - 1$ _______ critical points at $x = -1, 4$, IP at $x = 3/2$

III $f(x) = x^3 - 6x^2 + 9x - 1$ _______ critical points at $x = -1, 4$, IP at $x = 3/2$

i $f(x) = 2x^3 - 9x^2 + 24x + 1$ _______ critical points NONE, IP at $x = 3/2$

ii $f(x) = 2x^3 - 15^2 + 42x - 1$ _______ critical points NONE, IP at $x = 5/2$

iii $f(x) = 2x^3 - 7x^2 + 16x + 1$ _______ critical points NONE, IP at $x = 7/6$

1 $f(x) = x^3 - 9x^2 + 27x + 1$ _______ critical points at $x = -3, -3$, IP at $x = -3$

2 $f(x) = x^3 - 6x^2 + 12x - 2$ _______ critical points at $x = 2, 2$, IP at $x = 2$

3 $f(x) = x^3 + 12x^2 + 48x - 1$ _______ critical points at $x = -4, -4$, IP at $x = -4$

It is often claimed by students that for a cubic polynomial, *The inflection point is always half way between the critical points.*

1. Was that true for I, II, and III?

2. In what way was it true for 1, 2, and 3?

3. In what way was it true for i, ii, and iii?

4. Can you determine on what level, and why, it is true for every cubic polynomial?