Math 2414 Take Home Exam Five

Group Members Names

1. Label the following are always true, sometimes true, or never true. Either provide a proof or for the sometimes, give two examples, one showing the truth, one showing the non-truth.

   a If $\{r_n\} > 0$ for all n but $\{r_n\}$ is not eventually decreasing, then $\sum r_n$ diverges.

   b If $\sum |r_n|$ converges, then $\sum r_n$ converges.

   c If $\sum r_n$ converges, then $\sum |r_n|$ converges.

   d If $\{r_n\} > 0$ for all n and $\{r_n\} \to 0$, then $\sum (-1)^n r_n$ converges.

   e If $\{r_n\} > 0$ for all n and $\sum r_n$ converges, then $\sum (-1)^n r_n$ converges.

   f If $\sum r_n$ converges, then $\sum \frac{(-1)^n r_n}{n}$ converges.

   g $\sum r_n$ converges, then $\sum r_n^2$ converges.

   h If $\{r_n\} > 0$ for all n and $\sum r_n$ converges, then $\sum r_n^2$ converges.

   i If $R_n \leq E_n$ and $\sum E_n$ converges, then $\sum R_n$ converges.

   j If $0 \leq R_n \leq E_n$ and $\sum E_n$ converges, then $\sum R_n$ converges.

   k If $\sum R_n$ converges and the sequence $\{E_n/R_n\}$ converges, then $\sum E_n$ converges. (Assume $R_n > 0$ and $E_n > 0$)

   l If $\sum R_n$ converges and the sequence $\{R_n/E_n\}$ converges, then $\sum E_n$ converges. (Assume $R_n > 0$ and $E_n > 0$)

   m If $\lim_{n\to\infty} (r_{n+1}/r_n) = L$ and $L < 1$, then $\sum r_n$ converges.

2. First, show that the following series diverges, and explain why this does NOT violate the alternating series test.

$$\frac{1}{3} + \frac{-2}{5} + \frac{3}{7} + \frac{-4}{9} + \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{2k+1}$$

   Now show that the ratio test fails for the series below. Does the series converge, and if so, to what?

$$\frac{1}{2} + \frac{-1}{2} + \frac{1}{4} + \frac{-1}{4} + \frac{1}{8} + \frac{-1}{8} + \cdots$$

   Now show that the alternating series test fails for the series below. Does the series converge, and if so, to what?

$$\frac{1}{3} + \frac{-1}{3} + \frac{1}{2} + \frac{-1}{2} + \frac{1}{5} + \frac{-1}{5} + \frac{1}{4} + \frac{-1}{4} + \frac{1}{7} + \frac{-1}{7} + \frac{1}{6} + \frac{-1}{6} + \frac{1}{9} + \frac{-1}{9} + \frac{1}{8} + \frac{-1}{8} + \cdots$$

   Use the fact given below to show that $\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} \qquad -1 < x < 1$$

   Now show that the following rearrangement of the terms in the above series has the given sum:

$$\frac{3}{2} \ln 2 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

   Explain how this does NOT prove $\ln 2 = \frac{3}{2} \ln 2$

3. For the sequence R_n defined below, (1) Does $\lim_{n\to\infty} R_n$ exist? If so, find it. (2) Does $\sum R_n$ converge? Explain. (3) Does $\sum (-1)^n R_n$ converge? Explain.

$$R_n = \frac{1+2^n}{1+3*2^n}$$

Now for the sequence E_n defined below, (1) Does $\lim_{n\to\infty} R_n$ exist? If so, find it. (2) Does $\sum R_n$ converge? Explain. (3) Does $\sum (-1)^n R_n$ converge? Explain.

$$R_n = \frac{1+n2^n}{1+n^2*2^n}$$

4. Consider the integral given below. Explain why this integral converges. (In fact, it converges to $\pi^2/6$

$$\int_0^\infty \frac{xe^{-x}}{1-e^{-x}}$$

a Use the substitution $u = 1 - e^{-x}$ on the integral.

b After finishing the substitution, use a power series for the function in the integral, integrate term by term to get a series of numbers.

c About how many terms of this series would you ahve to add up to get a sum that is within 0.0001 of the correct value of the integral?

5. What is the coefficient of x^{100} in the power series for e^{2x} about $x = 0$?

Now evaluate $f^{(100)}(0)$ for the function f, where

$$f(x) = \begin{cases} \frac{1-\cos x}{x^2} & \text{for } x \neq 0 \\ \frac{1}{2} & \text{for } x = 0 \end{cases}$$

6. For each of the following series, find the sum.

a $x + x^3 + \frac{x^5}{2!} + \frac{x^7}{3!} + \cdots$

b $2 - 3*2x + 4*3x^2 - 5*4x^3 + \cdots$

c $\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \frac{x^8}{8} + \cdots$

d $1 - \frac{3x^2}{2!} + \frac{5x^4}{4!} - \frac{7x^6}{6!} + \cdots$

I $\sum_{k=1}^{\infty} \frac{k}{2^k}$

II $\sum_{k=1}^{\infty} \frac{1}{k2^k}$

III $\sum_{k=1}^{\infty} \frac{2^k}{(k+1)!}$