Math 2414 Take Home Exam Five ANSWERS

1. Label the following are always true, sometimes true, or never true. Either provide a proof or for the sometimes, give two examples, one showing the truth, one showing the non-truth.

   *** Make sure you understand the answers here, and be able to make a similar argument on the exam.

   a If $\{r_n\} > 0$ for all n but $\{r_n\}$ is not eventually decreasing, then $\sum r_n$ diverges. SOMETIMES. $r_n = 1 \quad \forall n$ clearly fits the bill but the series will diverge, but $r_n = 1/2^n$ for n odd and $= 1/3^n$ for n even, for example, will converge.

   b If $\sum |r_n|$ converges, then $\sum r_n$ converges. TRUE, this is the definition of absolute convergence.

   c If $\sum r_n$ converges, then $\sum |r_n|$ converges. SOMETIMES. $\sum (-1)^n/2^n$ works, but $\sum (-1)^n/n$ does not work.

   d If $\{r_n\} > 0$ for all n and $\{r_n\} \to 0$, then $\sum (-1)^n r_n$ converges. SOMETIMES $r_n = 1/n$ will be true, but $r_n = 1/n$ for n odd and $= 1/n^2$ for n even will not work.

   e If $\{r_n\} > 0$ for all n and $\sum r_n$ converges, then $\sum (-1)^n r_n$ converges. TRUE. This is basically absolute convergence gives conditional convergence.

   f If $\sum r_n$ converges, then $\sum \frac{(-1)^n r_n}{n}$ converges. SOMETIMES $r_n = 1/2^n$ makes this true, but $r_n = (-1)^n/(\ln n)$ will not work. Notice there is no assumption that $\sum r_n$ is NOT an alternating series to start with.

   g $\sum r_n$ converges, then $\sum r_n^2$ converges. SOMETIMES If $r_n > 0$ for all terms, then yes, as the squaring only makes it smaller, say $r_n = 1/2^n$. However, if $r_n = (-1)^n/sqrtn$ say, then the statement is not true.

   h If $\{r_n\} > 0$ for all n and $\sum r_n$ converges, then $\sum r_n^2$ converges. TRUE. As mentioned above, for n large enough, $r_n < 1$ so $0 \leq r_n^2 < r_n$ so we will get convergence by the comparison test.

   i If $R_n \leq E_n$ and $\sum E_n$ converges, then $\sum R_n$ converges. SOMETIMES, so for example $E_n = 1/2^n, R_n = 1/3^n$ then yes, but $E_n = 0, R_n = -1$ gives no.

   j If $0 \leq R_n \leq E_n$ and $\sum E_n$ converges, then $\sum R_n$ converges. TRUE. Compare with above. Now there is no way for R_n not to get squeeze, so to speak, to converge.

   k If $\sum R_n$ converges and the sequence $\{E_n/R_n\}$ converges, then $\sum E_n$ converges. (Assume $R_n > 0$ and $E_n > 0$) TRUE, the limit comparison test applies here.

   l If $\sum R_n$ converges and the sequence $\{R_n/E_n\}$ converges, then $\sum E_n$ converges. (Assume $R_n > 0$ and $E_n > 0$) SOMETIMES. $R_N = E_n = 1/n^2$ works, but $R_n = 1/n^2$ and $E_n = 1/n$ will not work. (Zero in the limit comparison test depends upon what is in the denominator remember)

   m If $\lim_{n\to\infty} (r_{n+1}/r_n) = L$ and $L < 1$, then $\sum r_n$ converges. SOMETIMES, $r_n = 1/2^n$ will work, but $r_n = (-1)^n 2^n$ will not work, hence the need for the absolute value in the ratio test.

2. First, show that the following series diverges, and explain why this does NOT violate the alternating series test.

$$\frac{1}{3}+\frac{-2}{5}+\frac{3}{7}+\frac{-4}{9}+\cdots=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k}{2k+1}$$

** The Alternating Series Test needs the sequence to go to zero, be positive, and decreasing in the tail. But the sequence of terms given above fails the limit part, as $\lim\limits_{k\to\infty}\frac{k}{2k+1}=1$by L'Hopitals if nothing else so by the Divergence Theorem, the series above will diverge.
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Now show that the ratio test fails for the series below. Does the series converge, and if so, to what?

$$\frac{1}{2}+\frac{-1}{2}+\frac{1}{4}+\frac{-1}{4}+\frac{1}{8}+\frac{-1}{8}+\cdots$$

*** If we group the terms in pairs, each pair telescopes to zero, so the series sums to zero. However, if we apply the ratio test, we either get 0 or 1, depending upon which pairs of terms we take, so the ratio test fails.
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Now show that the alternating series test fails for the series below. Does the series converge, and if so, to what?

$$\frac{1}{3}+\frac{-1}{3}+\frac{1}{2}+\frac{-1}{2}+\frac{1}{5}+\frac{-1}{5}+\frac{1}{4}+\frac{-1}{4}+\frac{1}{7}+\frac{-1}{7}+\frac{1}{6}+\frac{-1}{6}+\frac{1}{9}+\frac{-1}{9}+\frac{1}{8}+\frac{-1}{8}+\cdots$$

*** The Alternating Series Test needs the terms to be decreasing in the tail, which will never happen for this series, so the test fails. However, again, if we pair the terms together, they pair up to zero, and so the series has a sum of 0.
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Use the fact given to show that $\ln 2=\sum\limits_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$ $\qquad \ln(1+x)=\sum\limits_{k=1}^{\infty}\frac{(-1)^{k+1}x^k}{k}$ $\qquad -1<x<1$

*** If $0<x<1$ then $\sum\limits_{k=1}^{\infty}\frac{(-1)^{k+1}x^k}{k}$ is an alternating series whose sum is known. Thus, the difference between the partial sum and the sum of the series is bounded by the next term. (We discussed this idea with remainders and how far out to go in the series in class on Monday) So we have:

$$\left|\ln(1+x)-\sum_{k=1}^{N}\frac{(-1)^{k+1}x^k}{k}\right|=\left|\sum_{k=N+1}^{\infty}\frac{(-1)^{k+1}x^k}{k}\right|\leq\frac{|x|^{N+1}}{N+1}\leq\frac{1}{N+1}\quad\text{So this gives}\quad\left|\ln 2-\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\right|\leq\frac{1}{n+1}$$

So now if we first let $x\to 1^-$ we get the inequality below. In that inequality, if we now let $n\to\infty$, we get the desired result:

Now show that the following rearrangement of the terms in the above series has the given sum: $\frac{3}{2}\ln 2=1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots$
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$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}-\tfrac{1}{8}\cdots=\ln 2$$

$$+0+\tfrac{1}{2}+0-\tfrac{1}{4}+0+\tfrac{1}{6}+0-\tfrac{1}{8}+0+\cdots=\tfrac{1}{2}\ln 2$$

$$\Rightarrow 1+0+\tfrac{1}{3}+-\tfrac{1}{2}+\tfrac{1}{5}+0+\tfrac{1}{7}-\tfrac{1}{4}+\cdots=\tfrac{3}{2}\ln 2$$

Explain how this does NOT prove $\ln 2=\frac{3}{2}\ln 2$

*** It is the major 'problem' with infinite things, they do not follow all the rules we would like. In fact, it has been known for over 100 years that you can rearrange the terms in a converging alternating series to get any sum you wish.

3. For the sequence R_n defined below, (1) Does $\lim_{n\to\infty} R_n$ exist? If so, find it. (2) Does $\sum R_n$ converge? Explain. (3) Does $\sum(-1)^n R_n$ converge? Explain.

$$R_n = \frac{1+2^n}{1+3*2^n}$$

$$\text{(1) } \lim_{n\to\infty} \frac{1+2^n}{1+3*2^n} = \frac{\infty}{\infty} \overset{LH}{\to} \lim_{n\to\infty} \frac{\ln(2)2^n}{3*\ln(2)2^n} = \frac{1}{3}$$

(2) So the Divergence Theorem kicks in based upon the above limit and says that the series $\sum_n^\infty R_n$ diverges.

(3) Same reason as (2) above, the Alternating Series Test needs the sequence to go to zero, which it does not, so $\sum_n^\infty (-1)^n R_n$ diverges.

Now for the sequence E_n defined below, (1) Does $\lim_{n\to\infty} R_n$ exist? If so, find it. (2) Does $\sum R_n$ converge? Explain. (3) Does $\sum(-1)^n R_n$ converge? Explain.

$$R_n = \frac{1+n2^n}{1+n^2*2^n}$$

$$\text{(1) } \lim_{n\to\infty} \frac{1+n2^n}{1+n^2*2^n} = \frac{\infty}{\infty} \overset{LH}{\to} \lim_{n\to\infty} \frac{n\ln(2)2^n+2^n}{n^2\ln(2)2^n+2n2^n} = \lim_{n\to\infty} \frac{n\ln(2)+1}{n^2\ln(2)+2n} = \frac{\infty}{\infty} \overset{LH}{\to} \lim_{n\to\infty} \frac{\ln 2}{2n\ln(2)+2} \to 0$$

(2) No, the series will diverge. Compare with $\sum 1/n$.

(3) By (1) above, the limit of the sequence is zero, the terms of the sequence are positive and decreasing, so by the Alternating Series Test, $\sum(-1)^n R_n$ converges.

4. Consider the integral given below. Explain why this integral converges. $\int_0^\infty \frac{xe^{-x}}{1-e^{-x}} = \pi^2/6$

$$f(x) = \frac{xe^{-x}}{1-e^{-x}} = \frac{x}{e^x-1} \quad \lim_{x\to 0} f(x) = 1 \text{ (by L'Hopital's) so the integral is fine at } x=1$$

$$\sum \frac{ne^{-n}}{1-e^{-n}} \text{ converges by limit comparison with } \sum \frac{n}{e^n}$$

So there is no problem, the integral will converge.

a Use the substitution $u = 1-e^{-x}$ on the integral. After finishing the substitution, use a power series for the function in the integral, integrate term by term to get a series of numbers.

$$u = 1-e^{-x} \Rightarrow x = -\ln(1-u) \text{ so } \int_0^\infty \frac{xe^{-x}}{1-e^{-x}} = \int_{u=0}^{u=1} \frac{-\ln(1-u)}{u}du = \int_{u=0}^{u=1} \frac{-u-\frac{u^2}{2}-\frac{u^3}{3}-\cdots}{u} =$$

$$\int_{u=0}^{u=1} \sum_{k=0}^\infty \frac{u^k}{k+1}du = \sum_{k=0}^\infty \frac{u^{k+1}}{(k+1)^2}\bigg|_0^1 = \sum_{k=0}^\infty \frac{1}{k^2} = \frac{\pi^2}{6}$$

b About how many terms of this series would you have to add up to get a sum that is within 0.0001 of the correct value of the integral?

$$\int_n^\infty \frac{dx}{x^2} = \frac{1}{n}, \text{ so roughly 10,000 terms}$$

5. What is the coefficient of x^{100} in the power series for e^{2x} about $x=0$?

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} \Rightarrow \frac{2^{100}}{100!} \text{ is the coefficient of } x^{100}$$

Now evaluate $f^{(100)}(0)$ for the function f, where

$$f(x) = \begin{cases} \frac{1-\cos x}{x^2} & \text{for } x \neq 0 \\ \frac{1}{2} & \text{for } x = 0 \end{cases}$$

If we take the series expansion for $\cos x$, then we get

$$f(x) = \frac{1-\cos x}{x^2} = \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots}{x^2} = \frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+2)!} =$$

$$\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \cdots - \frac{x^{98}}{100!} + \frac{x^{100}}{102!} - \frac{x^{102}}{104!} + \cdots \Rightarrow f^{(100)}(x) = \frac{100!}{102!} - \frac{102!x^2}{2!104!} + \frac{104!x^4}{4!106!x^4} + \cdots \Rightarrow f^{(100)}(0) = \frac{100!}{102!} = \frac{1}{102*101}$$

6. For each of the following series, find the sum.

a $x + x^3 + \frac{x^5}{2!} + \frac{x^7}{3!} + \cdots = x(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots) = x(1 + (x)^2 + \frac{(x^2)^2}{2!} + \frac{(x^3)^2}{3!} + \cdots) = xe^{x^2}$

b $2 - 3*2x + 4*3x^2 - 5*4x^3 + \cdots = \sum_{k=0}^{\infty}((k+2)(k+1)x^k)$ take two anti-derivatives to get $\sum_{0}^{\infty} x^{k+2}$ So take two derivatives of $\frac{1}{1-x}$ then replace x with $-x$ to get $\frac{2}{(1+x)^3}$

c $\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \frac{x^8}{8} + \cdots$ Take a derivative to get $x + x^3 + x^5 + \cdots = x(1 + x^2 + x^4 + \cdots) = x(1 + (x)^2 + (x^2)^2 + (x^3)^2 + \cdots$ So take $\frac{1}{1-x}$, replace x with x^2, multiply by x, then integrate to get $(-1/2)\ln(1-x^2)$

d $1 - \frac{3x^2}{2!} + \frac{5x^4}{4!} - \frac{7x^6}{6!} + \cdots$ Integrate, then factor to get $x(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots) = x\cos x$ so $\frac{d}{dx}(x\cos x) = \cos x - x\sin x$

I $\sum_{k=1}^{\infty} \frac{k}{2^k} = \sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k = \sum_{k=1}^{\infty} kx^k$ for $x = 1/2$ Factor out an x, integrate, and we get $\frac{1}{1-x}$ So $\frac{x}{(1-x)^2}$ when $x = 1/2$ which gives 2.

II $\sum_{k=1}^{\infty} \frac{1}{k2^k} = \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{1}{2}\right)^k = \sum_{k=1}^{\infty} \frac{1}{k}x^k$ when $x = 1/2$ So take derivative to get the series for $\frac{1}{1-x}$ So integrate $\frac{1}{1-x}$ and then plug in 1/2 to get $-\ln(1-1/2) = \ln(2)$

III $\sum_{k=1}^{\infty} \frac{2^k}{(k+1)!} = \sum_{k=1}^{\infty} \frac{x^k}{(k+1)!}$ when $x = 2$. $\sum_{k=1}^{\infty} \frac{x^k}{(k+1)!} = \frac{1}{x}\sum_{k=1}^{\infty} \frac{x^{k+1}}{(k+1)!} = \frac{1}{x}(e^x - 1)$ So when $x = 2$ we get $\frac{e^2-1}{2}$