Math 2414 Take home Questions Exam 4 Due at the end of the review day

1. For each of the five definite integrals given below, explain which ones are improper and why. For each of them, compute the value of the definite integral like you would on the in class part of the exam.

   I $\int_1^\infty \frac{\sin x}{x}\,dx \approx 0.624713$ You should have easily seen that it is improper, but the actual evaluation of this *Dirichlet Integral* is well beyond the scope of this class.

   II $\int_4^5 \frac{1}{x}\,dx = \ln(5/4)$ No problems here.

   III $\int_3^4 \frac{dx}{\sin x} = \int\limits_3^4 \csc x\,dx$ There is a problem at π, where $\csc x$ is undefined, so this is improper.

   $$\lim_{E\to\pi^-}\int\limits_3^E \csc x\,dx + \lim_{B\to\pi^+}\int\limits_B^4 \csc x\,dx = \ln|\infty+\infty| - \ln|\csc 3+\cot 3| + \ln|\csc 4+\cot 4| - \ln|\csc\pi+\cot\pi|$$

   This diverges or is unbounded.

   IV $\int_{-3}^3 x^{-1/3}\,dx$ This has a problem at $x=0$, an asymptote, so this is improper.

   $$\lim_{E\to 0^-}\int\limits_{-3}^E x^{-1/3}\,dx + \lim_{B\to 0^+}\int\limits_{-3}^E x^{-1/3}\,dx = \left.\frac{3x^{2/3}}{2}\right|_{-3}^E + \left.\frac{3x^{2/3}}{2}\right|_B^3 = 0$$

   V $\int_{-10}^{10} f(x)dx$ where $f(x)=2+1/x,\quad -10\le x<-1 \qquad f(x)=\frac{1}{x+2},\quad -1\le x\le 10$ NOT improper, as there is no ∞ and no asymptote.

   $$\int\limits_{-10}^{-1}(2+1/x)dx + \int\limits_{-1}^{10}\frac{1}{x+2}dx = (2x+\ln|x|)|_{-10}^{-1} + \ln|x+2||_{-1}^{10} = 18+\ln(6/5)$$

2. Write each of the following in summation notation, then write the definite integral that goes with it.

(I) $\lim\limits_{n\to\infty}\frac{1}{n}\left[(\frac{1}{n})^3+(\frac{2}{n})^3+\cdots+(\frac{n}{n})^3\right] = \lim\limits_{n\to\infty}\sum\limits_{k=1}^n \frac{1}{n}\left(\frac{k}{n}\right)^3 = \int\limits_0^1 x^3\,dx$

(II) $\lim\limits_{n\to\infty}\frac{1}{n}\left[(\frac{1^5}{n^5})+(\frac{2^5}{n^5})+\cdots+(\frac{n^5}{n^5})\right] = \lim\limits_{n\to\infty}\sum\limits_{k=1}^n \frac{1}{n}\left(\frac{k}{n}\right)^5 = \int\limits_0^1 x^5\,dx$

(III) $\lim\limits_{n\to\infty}\frac{7}{n}\left[(\frac{n+7}{n})^3+(\frac{n+14}{n})^3+(\frac{n+21}{n})^3+(\frac{n+7n}{n})^3\right] = \lim\limits_{n\to\infty}\sum\limits_{k=1}^n \frac{7}{n}\left(1+\frac{7k}{n}\right)^3 = \int\limits_1^8 x^3\,dx$

(IV) $\lim\limits_{n\to\infty}\frac{42}{n}\left[(\frac{2n+42}{n})^5+(\frac{2n+84}{n})^5+(\frac{2n+126}{n})^5+(\frac{2n+42n}{n})^5\right] = \lim\limits_{n\to\infty}\sum\limits_{k=1}^n \frac{1}{n}\left(2+\frac{42k}{n}\right)^5 = \int\limits_2^{44} x^5\,dx$

(R_1) $\lim\limits_{n\to\infty}\frac{1}{n}\left[\frac{n}{n+1}+\frac{n}{n+2}+\frac{n}{n+3}+\frac{n}{n+4}+\cdots+\frac{n}{n+n}\right] = \lim\limits_{n\to\infty}\frac{1}{n}\left[\frac{1}{\frac{n+1}{n}}+\frac{1}{\frac{n+2}{n}}+\frac{1}{\frac{n+3}{n}}+\frac{1}{\frac{n+4}{n}}+\cdots+\frac{1}{\frac{n+n}{n}}\right] =$

$\lim\limits_{n\to\infty}\frac{1}{n}\left[\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\frac{1}{1+\frac{3}{n}}+\frac{1}{1+\frac{4}{n}}+\cdots+\frac{1}{1+\frac{n}{n}}\right] = \lim\limits_{n\to\infty}\sum\limits_{k=1}^n\frac{1}{n}\left(\frac{1}{1+\frac{k}{n}}\right) = \int\limits_1^2\frac{1}{x}dx$

(R_2) From looking at R_1, it seems this function is similar to that, but with some differences. For example, there is no $1/n$ out front. But the inside terms are all similar to the inside terms from R_1, suggesting we are starting at 1. So it would seem the interval is $[1,n+1]\to\Delta x=\frac{n+1-1}{n}=1$ But that means the interval is growing as n gets bigger, so the integral must be improper.

$\lim\limits_{n\to\infty}\left[\frac{1}{1+1}+\frac{1}{1+2}+\frac{1}{1+3}+\frac{1}{1+4}+\frac{1}{1+5}+\cdots+\frac{1}{2+n}\right] = \lim\limits_{n\to\infty}\sum\limits_{k=1}^n\frac{1}{n}\left(\frac{1}{1+\frac{k}{n}}\right) = \int\limits_1^\infty\frac{1}{x}dx$

(V) Now, having done the ones above, that gives us some insight into how to handle this one.

$\lim\limits_{n\to\infty}\frac{n+1}{n}\left[(\frac{n+1}{n})^3+(\frac{n+2}{n})^3+\cdots+(\frac{2n}{n})^3\right] = \lim\limits_{n\to\infty}(1+\frac{1}{n})\left[(\frac{n+1}{n})^3+(\frac{n+2}{n})^3+\cdots+(\frac{2n}{n})^3\right]$

$= \int\limits_1^\infty(1+x)^3\,dx + \int\limits_1^2(1+x)^3\,dx$

3. Find the sums of each of the following series. Make note of any patterns you see.

$$S_1 = \sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k+3}}\right) = \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{4}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}}\right) + \left(\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{6}}\right) + \cdots = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}$$

$$S_2 = \sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k+4}}\right) = \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{5}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{7}}\right) + \left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{8}}\right) + \cdots = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}}$$

$$S_3 = \sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k+5}}\right)$$
$$= \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{7}}\right) + \left(\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{8}}\right) + \left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{9}}\right) + \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{10}}\right) + \cdots$$
$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}}$$

$$S_4 = \sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k+6}}\right)$$
$$= \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{7}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{8}}\right) + \left(\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{9}}\right) + \left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{10}}\right) + \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{11}}\right) + \left(\frac{1}{\sqrt{7}} - \frac{1}{\sqrt{12}}\right) + \cdots$$
$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}}$$

4. Find the sums of each of the following series. Make note of any patterns you see.

Partial fractions come in handy here, and then the series telescopes.

$$S_1 = \sum_{k=1}^{\infty} \frac{1}{k^2+2k} = \sum_{k=1}^{\infty} \left[\frac{1/2}{k} + \frac{-1/2}{k+2}\right] = \left(\frac{1/2}{1} + \frac{-1/2}{3}\right) + \left(\frac{1/2}{2} + \frac{-1/2}{4}\right) + \left(\frac{1/2}{3} + \frac{-1/2}{5}\right) \cdots$$

$$= (1/2)(1+1/2) = (1/2)\sum_{r=1}^{2} 1/r = 3/4$$

$$S_2 = \sum_{k=1}^{\infty} \frac{1}{k^2+3k} = \sum_{k=1}^{\infty} \left[\frac{1/3}{k} + \frac{-1/3}{k+3}\right] =$$

$$\left(\frac{1/3}{1} + \frac{-1/3}{4}\right) + \left(\frac{1/3}{2} + \frac{-1/3}{5}\right) + \left(\frac{1/3}{3} + \frac{-1/3}{6}\right) + \left(\frac{1/3}{4} + \frac{-1/3}{7}\right) + \cdots$$

$$= (1/3)(1+1/2+1/3) = (1/3)\sum_{r=1}^{3} 1/r = 11/18$$

$$S_3 = \sum_{k=1}^{\infty} \frac{1}{k^2+5k} = \sum_{k=1}^{\infty} \left[\frac{1/5}{k} + \frac{-1/5}{k+5}\right]$$

$$= \left(\frac{1/5}{1} + \frac{-1/5}{6}\right) + \left(\frac{1/5}{2} + \frac{-1/5}{7}\right) + \left(\frac{1/5}{3} + \frac{-1/5}{8}\right) + \left(\frac{1/5}{4} + \frac{-1/5}{9}\right) + \left(\frac{1/5}{5} + \frac{-1/5}{10}\right) + \left(\frac{1/5}{6} + \frac{-1/5}{11}\right) + \cdots$$

$$= (1/5)(1+1/2+1/3+1/4+1/5) = (1/5)\sum_{r=1}^{5} 1/r = 137/180$$

$$S_4 = \sum_{k=1}^{\infty} \frac{1}{k^2+8k} = \sum_{k=1}^{\infty} \left[\frac{1/8}{k} + \frac{-1/8}{k+8}\right]$$

$$= \left(\frac{1/8}{1} + \frac{-1/8}{9}\right) + \left(\frac{1/8}{2} + \frac{-1/8}{10}\right) + \left(\frac{1/8}{3} + \frac{-1/8}{11}\right) + \left(\frac{1/8}{4} + \frac{-1/8}{12}\right) +$$

$$\left(\frac{1/8}{5} + \frac{-1/8}{13}\right) + \left(\frac{1/8}{6} + \frac{-1/8}{14}\right) + \left(\frac{1/8}{7} + \frac{-1/8}{15}\right) + \left(\frac{1/8}{9} + \frac{-1/8}{16}\right) + \cdots$$

$$= (1/8)(1+1/2+1/3+1/4+1/5+1/6+1/7+1/8) = (1/8)\sum_{r=1}^{8} 1/r = 761/840$$

5. Label the following are always true, sometimes true, or never true. Either provide a proof or for the sometimes, give two examples, one showing the truth, one showing the non-truth.

*** Make sure you understand the answers here, and be able to make a similar arguement on the exam.

a If $\sum r_n$ converges, then $r_n \to 0$. TRUE - direct result of the Divergence Theorem.

b If $r_n \to 0$, then $\sum r_n$ converges. SOMETIMES, $\sum 1/n$, say, shows why this will not alwyas work, but $\sum 1/2^n$ say, will work.

c $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ whenever $x \neq 1$. SOMETIMES If $|x| < 1$ then it is true, but otherwise we have a geometric series with a ratio whose absolute value is larger than one, and will diverge.

d If $\sum E_n$ and $\sum B_n$ each diverge, then $\sum E_N + B_N$ diverges. SOMETIMES. $E_n = B_n = 1/n$ works, but $E_n = 1/n, B_n = -1/n$ is the series whith every term zero, and clearly converges.

e If $\sum E_n$ converges and $\sum B_n$ diverges, then $\sum E_N + B_N$ diverges. TRUE. A proof by contradiction: Suppose $\sum (E_n + B_n)$ did converge. Then $\sum B_n = \sum (E_n + B_n) - E_n$ would converge (sum of two convergant series).

f If $\{r_n\}$ is monotonic and bounded, then $\sum r_n$ converges. FALSE: Consider $r_n = 1 \quad \forall n$. The series diverges. The `sequence` converges here.

g If the partial sums of $\sum r_n$ are bounded, then $\sum r_n$ conveges.FALSE. Consider $r_n = (-1)^n$. The partial sums are either 0 or 1 (or 0 and -1 depending upon where n starts), so bounded, but series diverges.

h If $r_n \geq 0$ and the partial sums of $\sum r_n$ are bounded, then $\sum r_n$ converges. TRUE. Now we have a sequence of partial sums that is bounded, and increasing (since now all the terms of the sequence are non-negative), so a bounded monotonically increasing sequence of partial sums must converge.

i If $\sum E_n$ and $\sum B_n$ each converge, then $\sum (E_N * B_N) = (\sum E_N) * (\sum B_n)$. SOMETIMES. Actually, hardly ever. For one that does not work, try $E_n = B_n = 1/2^n$. A trivial case that works is $\sum E_n =$ anything that converges, $B_n = 0$. With some work, we can find various geometric series that work: $E_n = \sum 5/4^n, B_n = \sum 3(-1/2)^n$

** You should verify this one, and maybe think about how you might find others.....