Math 2414 Take home Questions set 2. Due at the end of class on the review day.

ANSWERS

1. For each of the following set ups, find the value of $R$ so that $x = R$ would divide the area of the region under the graph of $f$ in the first quadrant into equal areas.
   (a) $f_1(x) = 10 - x^2$
   (b) $f_2(x) = 10 - 2x^2$
   (c) $f_3(x) = 10 - 3x^2$
   (d) $f_4(x) = 10 - 4x^2$

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ANSWER: Notice you can either view this as finding $R$ so that $\int_0^R (10 - x^2)dx = \int_R^{\sqrt{10}} (10 - x^2)dx$, or you can set it up like I did on the next line.

For $f_1$ we want $\int_0^R (10 - x^2)dx = (1/2) \int_0^{\sqrt{10}} (10 - x^2)dx$ This leads to $10R - R^3/3 = (10\sqrt{10})/3$ Solving this with WA gives $R = 1.09825$ for the first quadrant.

$f_2$ we want $\int_0^R (10 - 2x^2)dx = (1/2) \int_0^{\sqrt{10/2}} (10 - 2x^2)dx$. This leads to $10R - R^3/3 = (10\sqrt{10/2})/3$ Solving this with WA gives $R = 0.759988$ for the first quadrant.

$f_3$ we want $\int_0^R (10 - 2x^2)dx = (1/2) \int_0^{\sqrt{10/3}} (10 - 2x^2)dx$. This leads to $10R - R^3/3 = (10\sqrt{10/3})/3$ Solving this with WA gives $R = 0.616387$ for the first quadrant.

$f_4$ we want $\int_0^R (10 - 2x^2)dx = (1/2) \int_0^{\sqrt{10/4}} (10 - 2x^2)dx$. This leads to $10R - R^3/3 = (10\sqrt{10/4})/3$ Solving this with WA gives $R = 0.532067$ for the first quadrant.

2. (30 points) For each of the four functions given below, (each of which is only defined on the interval $[-1,1]$) compute the surface area and volume that results when the function is revolved about the $x$-axis. What patterns do you see? Now do it again, but around the $y$-axis. Again, what patterns do you see?

$f_1(x) = x^2 - 1$ $\qquad$ $f_2(x) = x^4 - 1$ $\qquad$ $f_3(x) = x^6 - 1$ $\qquad$ $f_4(x) = x^8 - 1$

I $x$-axis volume

$$f_1(x) \to 2\int_0^1 \pi\left(x^2-1\right)^2\,dx = 16\pi/15 = 3.3510 \qquad f_2(x) \to 2\int_1^1 \pi\left(x^4-1\right)^2\,dx = 64\pi/45 = 4.4680$$

$$f_3(x) \to 2\int_1^1 \pi\left(x^6-1\right)^2\,dx = 144\pi/91 = 4.9713 \qquad f_4(x) \to 2\int_0^1 \pi\left(x^8-1\right)^2\,dx = 256\pi/153 = 5.2565$$

II $x$-axis surface area

$$f_1(x) \to 2\int_0^1 2\pi(x^2-1)\sqrt{1+[2x]^2}\,dx = 10.965 \qquad f_2(x) \to 2\int_0^1 2\pi(x^4-1)\sqrt{1+[4x^3]^2}\,dx = 13.236$$

$$f_3(x) \to 2\int_0^1 2\pi(x^6-1)\sqrt{1+[6x^5]^2}\,dx = 14.3919 \qquad f_4(x) \to 2\int_0^1 2\pi(x^8-1)\sqrt{1+[8x^7]^2}\,dx = 15.1069$$

III $y$-axis volume

$$f_1(x) \to \int_{-1}^0 \pi\left((y+1)^{1/2}\right)^2\,dy = \pi/2 = 1.5708 \qquad f_2(x) \to \int_{-1}^0 \pi\left((y+1)^{1/4}\right)^2\,dy = 2\pi/3 = 2.0944$$

$$f_3(x) \to \int_{-1}^0 \pi\left((y+1)^{1/6}\right)^2\,dy = 3\pi/4 = 2.3562 \qquad f_4(x) \to \int_{-1}^0 \pi\left((y+1)^{1/8}\right)^2\,dy = 4\pi/5 = 2.5133$$

IV $y$-axis surface area CLEARLY (?!?) there is a problem here, as the arc length formula (and hence surface area) assumes a continuous derivative on the interval, which we do not have here. But we can still set it up and get a numerical value, we just have to realize the limitations of the method. Clearly (?!?) the surface area is approaching $6\pi \approx 18.85$

$$f_1(x) \to \int_{-1}^0 2\pi(y+1)^{1/2}\sqrt{1+\left[\frac{1}{2}(y+1)^{1/2-1}\right]^2}\,dy = 5.3304$$

$$f_2(x) \to \int_{-1}^0 2\pi(y+1)^{1/4}\sqrt{1+\left[\frac{1}{4}(y+1)^{1/4-1}\right]^2}\,dy = 6.34067$$

$$f_3(x) \to \int_{-1}^0 2\pi(y+1)^{1/6}\sqrt{1+\left[\frac{1}{6}(y+1)^{1/6-1}\right]^2}\,dy = 6.90961$$

$$f_4(x) \to \int_{-1}^0 2\pi(y+1)^{1/8}\sqrt{1+\left[\frac{1}{8}(y+1)^{1/8-1}\right]^2}\,dy = 7.28155$$

3. (30 points) For each of the four functions given below, compute the surface area and volume that results when the function is revolved about the $x$-axis. What patterns do you see? Now do it again, but around the $y$-axis. Again, what patterns do you see?

$f_1 : y^2 + x^2 = 4^2$ $\quad$ $f_2 : y^4 + x^4 = 4^4$ $\quad$ $f_3 : y^6 + x^6 = 4^6$ $\quad$ $f_4 : y^8 + x^8 = 4^8$

Yes, all of the graphs are symmetric about both the $x$ and $y$ axis, so we can take advantage of that when setting up the integrals. This should also serve as a double check that we get the same answers going around the $x$-axis as we do around the $y$-axis.

I $x$-axis volume

$$f_1 \to 2\int_0^4 \pi\left(\sqrt{4^2 - x^2}\right)^2 dx = 268.083 \qquad f_2 \to 2\int_0^4 \pi\left(\sqrt{4^4 - x^4}\right)^2 dx = 351.464$$

$$f_3 \to 2\int_0^4 \pi\left(\sqrt{4^6 - x^6}\right)^2 dx = 375.901 \qquad f_4 \to 2\int_0^4 \pi\left(\sqrt{4^8 - x^8}\right)^2 dx = 386.148$$

II $x$-axis surface area CLEARLY (?!?) there is a problem here, as the arc length formula (and hence surface area) assumes a continuous derivative on the interval, which we do not have here. Clearly (?!?) the surface area is approaching $96\pi \approx 301.593$

$$f_1(x) \to 2\int_0^4 2\pi\sqrt{4^2 - x^2}\sqrt{1 + \left[\frac{1}{2}(4^2 - x^2)^{1/2-1}(-2x)\right]^2}\, dx = 201.06$$

$$f_2(x) \to 2\int_0^4 2\pi\sqrt[4]{4^4 - x^4}\sqrt{1 + \left[\frac{1}{4}(4^4 - x^4)^{1/4-1}(-4x^3)\right]^2}\, dx = 839.346$$

$$f_3(x) \to 2\int_0^4 2\pi\sqrt[6]{4^6 - x^6}\sqrt{1 + \left[\frac{1}{6}(4^6 - x^6)^{1/6-1}(-6x^5)\right]^2}\, dx = 3367.94$$

$$f_4(x) \to 2\int_0^4 2\pi\sqrt[8]{4^8 - x^8}\sqrt{1 + \left[\frac{1}{8}(4^8 - x^8)^{1/8-1}(-8x^7)\right]^2}\, dx = 13430.8$$

III $y$-axis volume

$$f_1(x) \to 2\int_0^4 \pi\left((4^2 - y^2)^{1/2}\right)^2 dy = 268.083 \qquad f_2(x) \to 2\int_0^4 \pi\left((4^4 - y^4)^{1/4}\right)^2 dy = 351.464$$

$$f_3(x) \to 2\int_0^4 \pi\left((4^6 - y^6)^{1/6}\right)^2 dy = 375.901 \qquad f_4(x) \to 2\int_0^4 \pi\left((4^8 - y^8)^{1/8}\right)^2 dy = 386.148$$

IV $y$-axis surface area CLEARLY (?!?) there is a problem here, as the arc length formula (and hence surface area) assumes a continuous derivative on the interval, which we do not have here. Clearly (?!?) the surface area is approaching $96\pi \approx 301.593$

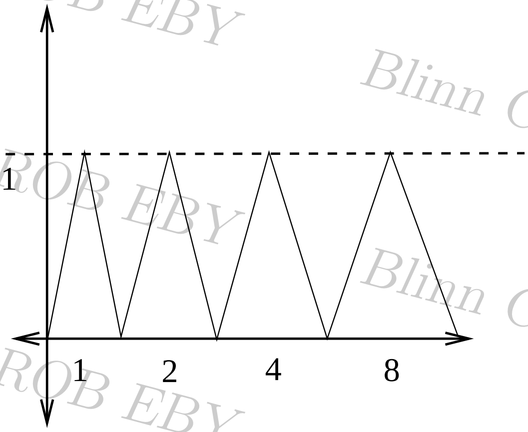
$$f_1(x) \to 2\int_0^4 2\pi(4^2 - y^2)^{1/2}\sqrt{1 + \left[\frac{1}{2}(4^2 - y^2)^{1/2-1}(-2y)\right]^2}\, dy = 201.06$$

$$f_2(x) \to 2\int_0^4 2\pi(4^4 - y^4)^{1/4}\sqrt{1 + \left[\frac{1}{4}(4^4 - y^4)^{1/4-1}(-4y^3)\right]^2}\, dy = 839.346$$

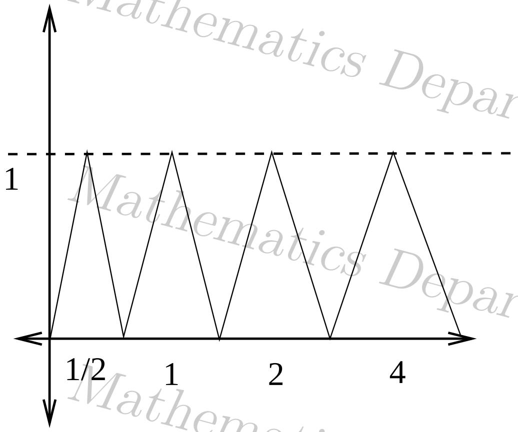
$$f_3(x) \to 2\int_0^4 2\pi(4^6 - y^6)^{1/6}\sqrt{1 + \left[\frac{1}{6}(4^6 - y^6)^{1/6-1}(-6y^5)\right]^2}\, dy = 3367.94$$

$$f_4(x) \to 2\int_0^4 2\pi(4^8 - y^8)^{1/8}\sqrt{1 + \left[\frac{1}{8}(4^8 - y^8)^{1/8-1}(-8y^7)\right]^2}\, dy = 13430.8$$

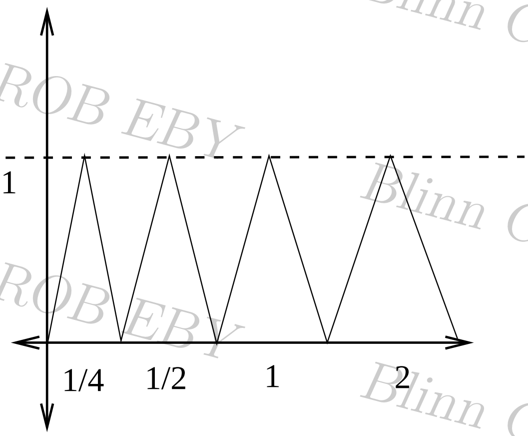
4. (30 points) For each graph below, compute the surface area and volume that results when the graph is revolved about the $x$-axis. What patterns do you notice? If you want, I give the function that generates the first graph below.



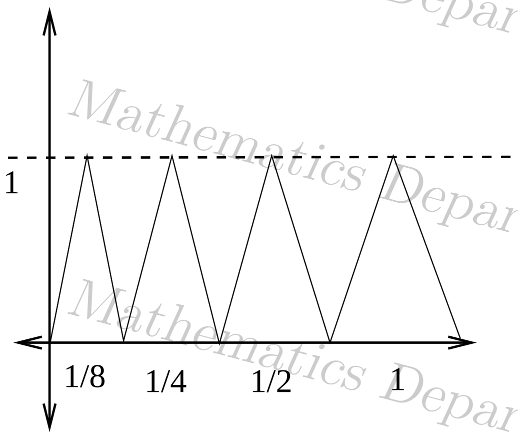

F1




F2




F3




F4

Clearly (?!?) each graph will form a set of stacked cones of various heights and therefore various slant heights. (The radius should always stay 1 for each one) So then we can use the volume and surface area formulas for cones here.

$$V = \frac{1}{3}\pi r^2 h \qquad SA = \pi r l = \pi r\sqrt{r^2+h^2}$$

| Graph | Volume | Surface Area |
|---|---|---|
| F1 | $\frac{2\pi}{3}(\frac{1}{2}+1+2+4) = 15.708$ | $2\pi\left(\frac{1}{2}\frac{\sqrt{5}}{2}+1\sqrt{2}+2\sqrt{5}+4\sqrt{17}\right) = 30.5422$ |
| F2 | $\frac{2\pi}{3}(\frac{1}{4}+\frac{1}{2}+1+2) = 7.854$ | $2\pi\left(\frac{1}{4}\frac{\sqrt{17}}{4}+\frac{1}{2}\frac{\sqrt{5}}{2}+1\sqrt{2}+2\sqrt{5}\right) = 17.9562$ |
| F3 | $\frac{2\pi}{3}(\frac{1}{8}+\frac{1}{4}+\frac{1}{2}+1) = 3.927$ | $2\pi\left(\frac{1}{8}\frac{\sqrt{65}}{8}+\frac{1}{4}\frac{\sqrt{17}}{4}+\frac{1}{2}\frac{\sqrt{5}}{2}+1\sqrt{2}\right) = 12.2819$ |
| F4 | $\frac{2\pi}{3}(\frac{1}{16}+\frac{1}{8}+\frac{1}{4}+\frac{1}{2}) = 1.963$ | $2\pi\left(\frac{1}{16}\frac{\sqrt{257}}{16}+\frac{1}{8}\frac{\sqrt{65}}{8}+\frac{1}{4}\frac{\sqrt{17}}{4}+\frac{1}{2}\frac{\sqrt{5}}{2}\right) = 9.8735$ |

5. For each of the functions given below, find the volume and surface area when the graph is revolved about the $y$-axis. Comment on the patterns you observe. The functions are only defined on $[0, 1]$. (It might help to graph each function)

- $f_1(x) = -x + 1 \quad 0 \le x \le 1$
- $f_2(x) = (-10/9)x + 1 \quad 0 \le x \le 9/10, \qquad f_2(x) = 0 \quad 9/10 \le x \le 1$
- $f_3(x) = (-9/8)x + 1 \quad 0 \le x \le 8/9, \qquad f_3(x) = 0 \quad 8/9 \le x \le 1$
- $f_4(x) = (-8/7)x + 1 \quad 0 \le x \le 7/8, \qquad f_4(x) = 0 \quad 7/8 \le x \le 1$
- $f_5(x) = (-7/6)x + 1 \quad 0 \le x \le 6/7, \qquad f_5(x) = 0 \quad 6/7 \le x \le 1$

So when I revolve each function about the $y$-axis, I get a cone and then a washer. (A top upside down if you wish) So we can again use the formulas for the cone like we did in the last problem.

$$V = \frac{1}{3}\pi r^2 h \qquad SA = \pi r l = \pi r\sqrt{r^2 + h^2}$$

| Graph | Volume | Surface Area |
|---|---|---|
| f1 | $\pi/3 = 1.047$ | $\pi\sqrt{2} = 4.4429$ |
| f2 | $\frac{1}{3}\pi(\frac{9}{10})^2 = 0.8482$ | $\pi(1)\sqrt{1 + (9/10)^2} + \pi(1^2 - (\frac{9}{10})^2) = 4.8234$ |
| f3 | $\frac{1}{3}\pi(\frac{8}{9})^2 = 0.8274$ | $\pi(1)\sqrt{1 + (8/9)^2} + \pi(1^2 - (\frac{8}{9})^2) = 5.0016$ |
| f4 | $\frac{1}{3}\pi(\frac{7}{8})^2 = 0.8018$ | $\pi(1)\sqrt{1 + (7/8)^2} + \pi(1^2 - (\frac{7}{8})^2) = 4.9108$ |
| f5 | $\frac{1}{3}\pi(\frac{6}{7})^2 = 0.7694$ | $\pi(1)\sqrt{1 + (6/7)^2} + \pi(1^2 - (\frac{6}{7})^2) = 4.9712$ |