Math 2414 Take Home Exam One ANSWERS

NOTE In order to fit better on pages, I changed the order of the problems presented.

1. Let $\boxplus(x)$ denote the greatest integer $\leq x, f(x) = \boxplus(x), g_2(x) = \boxplus(2x), g_3(x) = \boxplus(3x), g_4(x) = \boxplus(4x), h_2(x) = \boxplus(x/2), h_3(x) = \boxplus(x/3), h_4(x) = \boxplus(x/4)$. Find the following, and note what patterns you observe.

$$(1)\int_0^6 f(x)\,dx \quad (2)\int_0^6 g_2(x)\,dx \quad (3)\int_0^6 g_3(x)\,dx \quad (4)\int_0^6 g_4(x)\,dx \quad (5)\int_0^6 h_2(x)\,dx \quad (6)\int_0^6 h_3(x)\,dx \quad (7)\int_0^6 h_4(x)\,dx$$

ANSWER:

$$(1)\int_0^6 f(x)\,dx = 15 \quad (2)\int_0^6 g_2(x)\,dx = 33 \quad (3)\int_0^6 g_3(x)\,dx = 51 \quad (4)\int_0^6 g_4(x)\,dx = 69$$

$$(5)\int_0^6 h_2(x)\,dx = 6 \quad (6)\int_0^6 h_3(x)\,dx = 3 \quad (7)\int_0^6 h_4(x)\,dx = 2$$

There is a fairly obvious pattern of increases for the f and g functions. The pattern is rather hard to find for the f and h functions, but it does seem to have a lower limit.

2. Average value is discussed in the text on page 379 in section 5.4. Arrange the following intervals in the order for which the average value of $\sin x$ over the interval increases from smallest to largest. (Hint: a is last in the list)

a $0 \leq x \leq \pi$

b $\pi/2 \leq x \leq 3\pi/2$

c $\pi \leq x \leq 2\pi$

d $3.14 \leq x \leq 3.15$

In addition to finding the average value numerically, you should also explain your answer like you would if this showed up on the in class exam. (So you should argue from the shape and features of the graph of $\sin x$)

a $0 \leq x \leq \pi$ ______ $\dfrac{1}{\pi - 0}\displaystyle\int_0^\pi \sin x\,dx = 2/\pi$

b $\pi/2 \leq x \leq 3\pi/2$ ______ $\dfrac{1}{3\pi/2 - \pi/2}\displaystyle\int_{\pi/2}^{3\pi/2} \sin x\,dx = 0$

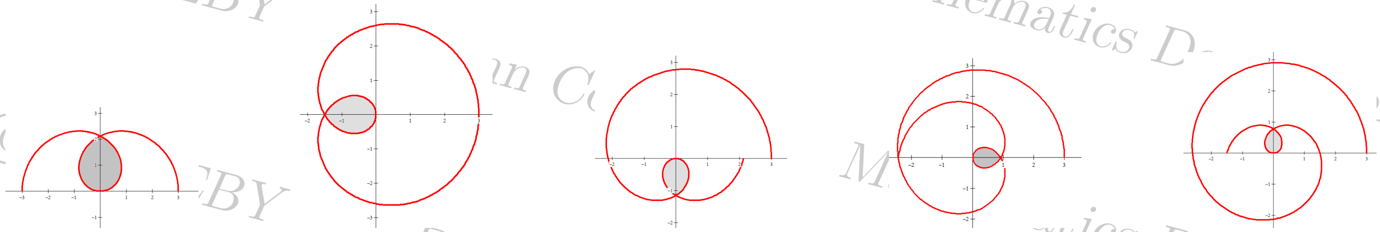
c $\pi \leq x \leq 2\pi$ ______ $\dfrac{1}{2\pi - \pi}\displaystyle\int_\pi^{2\pi} \sin x\,dx = -2/\pi$

d $3.14 \leq x \leq 3.15$ ______ $\dfrac{1}{3.15 - 3.14}\displaystyle\int_{3.14}^{3.15} \sin x\,dx = -0.003407$

The order is c, d, b, a. On $[0, \pi]$ sin is positive as the graph is above the axis. On $[\pi/2, 3\pi/2]$ the average value is exactly zero as the graph is equally above and below the graph. On $[\pi, 2\pi]$ the value is negative, equal in magnitude to the value for a. On $[3.14, 3.15]$ the value is close to zero since sin is close to zero at both points. Since $3.14 < \pi < 3.15$, sin changes value over the interval, and since π is closer to 3.14 than to 3.15 the average value is slightly negative.

3. Find the area of each *inner loop* (shaded) for each of the following polar curves. Make note of any patterns you see here. (All graphs except the first are on $[0, 4\pi]$, the first on $[0, 2\pi]$.

$r = 3\cos(t/2)$ $\quad r = 3\cos(t/3)$ $\quad r = 3\cos(t/4)$ $\quad r = 3\cos(t/5)$ $\quad r = 3\cos(t/6)$



ANSWER:

For $r = 3\cos(t/2)$ we have the following:

$$2\int_{\pi/2}^{\pi} \frac{1}{2}(3\cos(t/2))^2\,dt = \int_{\pi/2}^{\pi} 9\cos^2(t/2)\,dt = \int_{\pi/2}^{\pi} 9\left(\frac{1+\cos t}{2}\right)dt = \frac{9}{2}t + \frac{9}{2}\sin(t)\Big|_{\pi/2}^{\pi} = \frac{9\pi}{4} - \frac{9}{2} \approx 2.5686$$

In a similar fashion, for $r = 3\cos(t/3)$ we have

$$2\int_{\pi}^{3\pi/2} \frac{1}{2}(3\cos(t/3))^2\,dt = \frac{9\pi}{4} - \frac{27\sqrt{3}}{8} \approx 1.2229$$

In a similar fashion, for $r = 3\cos(t/4)$ we have

$$2\int_{3\pi/2}^{4\pi/2} \frac{1}{2}(3\cos(t/4))^2\,dt = \frac{9\pi}{4} - \frac{18\sqrt{2}}{4} \approx 0.70462$$

In a similar fashion, for $r = 3\cos(t/5)$ we have

$$2\int_{4\pi/2}^{5\pi/2} \frac{1}{2}(3\cos(t/5))^2\,dt = \frac{9\pi}{4} - \frac{45\sqrt{10-2\sqrt{5}}}{16} \approx 0.45600$$

In a similar fashion, for $r = 3\cos(t/6)$ we have

$$2\int_{\pi}^{3\pi/2} \frac{1}{2}(3\cos(t/3))^2\,dt = \frac{9\pi}{4} - \frac{27}{4} \approx 0.31858$$

There are several patterns to note here. (1) The shaded region keeps rotating around the orgin. (2) The areas are getting smaller. (3) The limits of integration progress in a nice pattern. (4) All answers involve $\frac{9\pi}{4}$

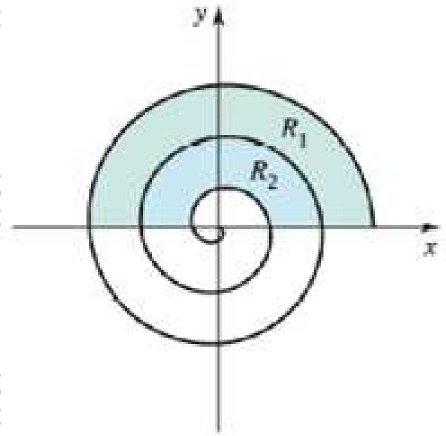
Indeed, if you think of the graph of $r = 3\cos(t/E)$, where E is a positive integer, then we actually have the area is as shown below:

$$2\int_{(E-1)\pi/2}^{E\pi/2} \frac{1}{2}\left(3\cos\left(\frac{t}{E}\right)\right)^2 dt = \int_{(E-1)\pi/2}^{E\pi/2}\left(9\cos^2\left(\frac{t}{E}\right)\right)dt = \frac{9\pi}{4} - \frac{9E}{4}\sin\left(\frac{E-1}{E}\pi\right)$$

Now it is simply a question of what happens as $E \to \infty$. Well, since all of my Calculus II students are really good at Calculus I, then I am sure they have no problem using L'Hopital's rule to evaluate the following limit:

$$\lim_{E\to\infty} \frac{9E}{4}\sin\left(\frac{E-1}{E}\pi\right) = \frac{9\pi}{4}$$

4. Let R_n be the region bounded by the nth and the $(n+1)$st turn of the spiral $r = e^{-\theta}$ in the first and second quadrants, for $\theta \geq 0$ (See the figure)



A Set up the integral(s) to find the area A_n of R_n for $n = 1, 2, 3, 4$.

B Now evaluate each of the integrals you found above.

C Compute each of the following: A_1/A_2, A_2/A_3, A_3/A_4.

ANSWER

PART A:

$$A_1 = \int_0^{\pi} \frac{1}{2}\left(r = e^{-\theta}\right)^2 \, d\theta - \int_{2\pi}^{3\pi} \frac{1}{2}\left(r = e^{-\theta}\right)^2 \, d\theta \qquad A_2 = \int_{2\pi}^{3\pi} \frac{1}{2}\left(r = e^{-\theta}\right)^2 \, d\theta - \int_{4\pi}^{5\pi} \frac{1}{2}\left(r = e^{-\theta}\right)^2 \, d\theta$$

$$A_3 = \int_{4\pi}^{5\pi} \frac{1}{2}\left(r = e^{-\theta}\right)^2 \, d\theta - \int_{5\pi}^{6\pi} \frac{1}{2}\left(r = e^{-\theta}\right)^2 \, d\theta \qquad A_4 = \int_{5\pi}^{6\pi} \frac{1}{2}\left(r = e^{-\theta}\right)^2 \, d\theta - \int_{7\pi}^{7\pi} \frac{1}{2}\left(r = e^{-\theta}\right)^2 \, d\theta$$

PART B

$$A_1 = \frac{-1}{4}e^{-2\pi} + \frac{1}{4}e^{0\pi} + \frac{1}{4}e^{-6\pi} + \frac{-1}{4}e^{-4\pi} \qquad A_2 = \frac{-1}{4}e^{-6\pi} + \frac{1}{4}e^{-4\pi} + \frac{1}{4}e^{-10\pi} + \frac{-1}{4}e^{-8\pi}$$

$$A_3 = \frac{-1}{4}e^{-10\pi} + \frac{1}{4}e^{-8\pi} + \frac{1}{4}e^{-14\pi} + \frac{-1}{4}e^{-12\pi} \qquad A_4 = \frac{-1}{4}e^{-14\pi} + \frac{1}{4}e^{-12\pi} + \frac{1}{4}e^{-18\pi} + \frac{-1}{4}e^{-16\pi}$$

PART C

$$\frac{A_1}{A_2} = e^{-4\pi} \qquad \frac{A_2}{A_3} = e^{-4\pi} \qquad \frac{A_3}{A_4} = e^{-4\pi}$$