

Problem 1 (20 points) Name and Section Number

You Can't Tune a Fish, But You Can Tuna Radio!

Problem 2 (50 points) In class we have discussed Taylor and power series. Now we will look at one application of such series. To start with, let us work with the function defined below, a step function. You can imagine this could represent the output of a device that switches from one state (-1) to another state (+1). First, create three graphs, with $step(x)$ on each graph, then f_1, f_2, f_3 one per graph. Label each graph and function. Also plot $step(x)$ and f_4 .

** The natural interval is $x = -\pi \dots \pi$ for these graphs.

$$step(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases} \quad f_1(x) = \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin 3x \quad f_4(x) = \frac{ie^{-ix}}{\pi} - \frac{ie^{ix}}{\pi} + \frac{ie^{-3ix}}{3\pi} - \frac{ie^{3ix}}{3\pi} + \frac{1}{2}$$

$$f_2(x) = \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin 3x + \frac{4}{5\pi} \sin 5x \quad f_3(x) = \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin 3x + \frac{4}{5\pi} \sin 5x + \frac{4}{7\pi} \sin 7x$$

Problem 3 (50 points) Produce three more graphs or pairs of functions, each on the interval $-\pi \leq x \leq \pi$. The pairs are given below.

$$g_1(x) = \frac{1}{2} + \frac{2 \cos x}{\pi} - \frac{2 \cos 2x}{2\pi} + \frac{2 \cos 3x}{3\pi} - \frac{2 \cos 4x}{4\pi} \quad square(x) = \begin{cases} 0 & -\pi < x < -\pi/2 \text{ or } \pi/2 < x < \pi \\ 1 & -\pi/2 < x < \pi/2 \\ 1/2 & x = \pm\pi/2 \end{cases}$$

$$sawtooth(x) = \begin{cases} x & -\pi < x < \pi \\ 0 & x = \pm\pi \end{cases} \quad g_2(x) = 2 \sin x - \sin 2x + \frac{2}{3} \sin 3x - \frac{1}{2} \sin 4x$$

$$triangle(x) = \begin{cases} \frac{x+\pi}{\pi} & -\pi \leq x \leq 0 \\ \frac{\pi-x}{\pi} & 0 \leq x \leq \pi \end{cases} \quad g_3(x) = \frac{1}{2} + \frac{4}{\pi^2} \cos x + \frac{4}{9\pi^2} \cos 3x + \frac{4}{25\pi^2} \cos 5x$$

Problem 4 (60 points) Okay, great, what the heck does that have to do with anything? Well, let us start with the easier part, and try to tune your radio to WEBY, 214 on the AM dial. In essence, you can think of tuning a radio as trying to pick out a function among the hundreds that are in the air. Well, basically, each station is broadcasting on a different function, so if there are E different stations, our radio picks up $f_1(t) + f_2(t) + \dots + f_E(t)$. We only want one station, 214, so how do we pick that one function out?

In general of course, if $f(t) = f_1(t) + f_2(t)$ we cannot find $f_1(t)$ or $f_2(t)$ if we only know $f(t)$. We can do this for radio (and similar signals) because the signals, or functions, are waves and are built from sines and cosines in a special way.

To start on that path, first compute $\int \cos nt \, dt$ and show that $\int_0^\pi \cos nt \, dt = 0, n \neq 0$.

Now show the following to be true. It may help you greatly to refresh your memory on the sum and difference formula for cos, and the half angle identities.

$$\int_0^\pi \sin(nt) \sin(mt) \, dt = 0 \quad \text{rm} \quad \int_0^\pi \cos(nt) \cos(mt) \, dt = 0 \quad n \neq \pm m$$

$$\int_0^\pi \sin^2 nt \, dt = \pi/2 \quad \int_0^\pi \cos^2 nt \, dt = \pi/2 \quad n = 1, 2, 3, \dots$$

Problem 5 (60 points) So how does this work? Well, let us look at a simplified case. When we hear 214 on the AM dial that is a designated frequency for the station, that is a function $\sin(nt)$. The station then sends out the signal $g(t)\sin(nt)$, where $g(t)$ is all the music, talk, etc. that we hear. Usually, $g(t)$ varies much more slowly than the $\sin(nt)$, since usually n is around one million. (Actually, sines and cosines are used, and the n does not have to be an integer, but we will simplify things a bit here)

Plot $t \sin(20t)$, $t^2 \sin(20t)$, $\sin(t) \sin(20t)$ on three different graphs. In each one, highlight the $t, t^2, \sin t$, the 'front' function. You should see that if $g(t)$ varies slowly compared to $\sin nt$, then the graph of $g(t) \sin nt$ is similar to a sine curve, but with amplitude determined by $g(t)$. This is *amplitude modulation*, or *AM signal*. The information ($g(t)$) is basically carried by the sine function, hence the $\sin nt$ is the *carrier wave*.

Problem 6 (60 points) So lets consider a basic example with a few stations coming in to the radio. Let us suppose the slowly varying signals (the voice, music, etc.) are $\sin t$ and $\sin 2t$. The chart below shows, for three stations, what they each want to send in terms of $\sin t$ and $\sin 2t$ and what the carrier wave is for each station.

| Station Name | sending function | carrier wave function | total signal sent |
|--------------|------------------------|-----------------------|-----------------------------------|
| S_1 | $3 \sin t - 2 \sin 2t$ | $\sin 8t$ | $(3 \sin t - 2 \sin 2t) \sin 8t$ |
| S_2 | $5 \sin t + 6 \sin 2t$ | $\sin 16t$ | $(5 \sin t + 6 \sin 2t) \sin 16t$ |
| S_3 | $4 \sin t + 7 \sin 2t$ | $\sin 24t$ | $(4 \sin t + 7 \sin 2t) \sin 24t$ |

The signals are all broadcast, which in mathematical terms means the functions are all super imposed on each other. So the signal reaching our radio is

$$f(t) = (3 \sin t - 2 \sin 2t) \sin 8t + (5 \sin t + 6 \sin 2t) \sin 16t + (4 \sin t + 7 \sin 2t) \sin 24t$$

Plot each station's signal and its sending function on its own graph, and also plot the signal our radio is receiving on its own graph. (So 4 graphs total here)

So how do we 'tune' to say, station S_2 and recover the signal? First, use the work you did in problem 4 above to write $f(t)$ in the following form: (you will need to determine the values of A, B, C, D, E, and F)

$$f(t) = A(\cos 7t - \cos 9t) - B(\cos 6t - \cos 10t) + C(\cos 15t - \cos 17t) + D(\cos 14t - \cos 18t) + E(\cos 23t - \cos 25t) + F(\cos 22t - \cos 26t)$$

Now, use the work you did in problem 4 above to show the following, and then multiply by -4π to recover the coefficients of $\sin t$ and $\sin 2t$ for station S_2 .

$$\int_0^\pi f(t) \cos 17t dt = \frac{-5\pi}{2} \quad \int_0^\pi f(t) \cos 18t dt = \frac{-6\pi}{2}$$

So now you should have the basic idea of how Fourier series are used in radio (and indeed many, many other places). Your work in problem 1 should have indicated how almost any function could be approximated by sines and cosines, and then your work in problem 6 gives an indication of how we can recover one signal from many.