MATH 2414 Guided Project Two

Problem 1 (20 points) Name and Section Number

You Can't Tune a Fish, But You Can Tuna Radio!

Problem 2 (50 points) In class we have discussed Taylor and power series. Now we will look at one application of such series. To start with, let us work with the function defined below, a step function. You can imagine this could represent the output of a device that switches from one state (-1) to another state (+1). First, create three graphs, with $step(x)$ on each graph, then f_1, f_2, f_3 one per graph. Label each graph and function. Also plot $step(x)$ and f_4.
** The natural interval is $x = -\pi \ldots pi$ for these graphs.

$$step(x) = \begin{cases} -1 & \text{if } \quad -\pi < x < 0 \\ 1 & \text{if } \quad 0 < x < \pi \end{cases} \quad f_1(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x \quad f_4(x) = \frac{ie^{-ix}}{\pi} - \frac{ie^{ix}}{\pi} + \frac{ie^{-3ix}}{3\pi} - \frac{ie^{3ix}}{3\pi} + \frac{1}{2}$$

$$f_2(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x \quad f_3(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \frac{4}{7\pi}\sin 7x$$

Problem 3 (50 points) Produce three more graphs or pairs of functions, each on the interval $-\pi \le x \le \pi$. The pairs are given below.

$$g_1(x) = \frac{1}{2} + \frac{2\cos x}{\pi} - \frac{2\cos 2x}{2\pi} + \frac{2\cos 3x}{3\pi} - \frac{2\cos 4x}{4\pi} \quad square(x) = \begin{cases} 0 & -\pi < x < -\pi/2 \text{ or } \pi/2 < x < \pi \\ 1 & -\pi/2 < x < \pi/2 \\ 1/2 & x = \pm\pi/2 \end{cases}$$

$$sawtooth(x) = \begin{cases} x & -\pi < x < pi \\ 0 & x = \pm\pi \end{cases} \qquad g_2(x) = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x$$

$$triangle(x) = \begin{cases} \frac{x+\pi}{\pi} & -\pi \le x \le 0 \\ \frac{\pi - x}{\pi} & 0 \le x \le \pi \end{cases} \qquad g_3(x) = \frac{1}{2} + \frac{4}{\pi^2}\cos x + \frac{4}{9\pi^2}\cos 3x + \frac{4}{25\pi^2}\cos 5x$$

Problem 4 (60 points) Okay, great, what the heck does that have to do with anything? Well, let us start with the easier part, and try to tune your radio to `WEBY, 214 on the AM dial`. In essence, you can think of tuning a radio as trying to pick out a function among the hundreds that are in the air. Well, basically, each station is broadcasting on a different function, so if there are E different stations, our radio picks up $f_1(t) + f_2(t) + \cdots + f_E(t)$. We only want one station, 214, so how do we pick that one function out?
In general of course, if $f(t) = f_1(t) + f_2(t)$ we cannot find $f_1(t)$ or $f_2(t)$ if we only know $f(t)$. We can do this for radio (and similar signals) because the signals, or functions, are waves and are built from sines and cosines in a special way.

To start on that path, first compute $\int \cos nt\, dt$ and show that $\int_0^\pi \cos nt\, dt = 0, n \neq 0$.

Now show the following to be true. It may help you grealy to refresh your memory on the sum and difference formula for cos, and the half angle identities.

$$\int_{)}^{\pi} \sin(nt)\sin(mt)dt = 0 \text{ rm } \int_0^\pi \cos(nt)\cos(mt)dt = 0 \quad n \neq \pm m$$

$$\int_0^\pi \sin^2 nt\, dt = \pi/2 \qquad \int_0^\pi \cos^2 nt\, dt = \pi/2 \qquad n = 1, 2, 3, \ldots$$

Problem 5 (60 points) So how does this work? Well, let us look at a simplified case. When we hear `214 on the AM dial` that is a designated frequency for the station, that is a function $\sin(nt)$. The station then sends out the signal $g(t)\sin(nt)$, where $g(t)$ is all the music, talk, etc. that we hear. Usually, $g(t)$ various much more slowly than the $\sin(nt)$, since usually n is around one million. (Actually, sines and cosines are used, and the n does not have to be an integer, but we will simplify things a bit here)
Plot $t\sin(20t), t^2\sin(20t), \sin(t)\sin(20t)$ on three different graphs. In each one, highlight the $t, t^2, \sin t$, the 'front' function. You should see that if $g(t)$ varies slowly compared to $\sin nt$, then the graph of $g(t)\sin nt$ is similar to a sine curve, but with amplitude determined by $g(t)$. This is *amplitude modulation*, or *AM signal*. The information ($g(t)$) is basically carried by the sine function, hence the $\sin nt$ is the *carrier wave*.

Problem 6 (60 points) So lets consider a basic example with a few stations coming in to the radio. Let us suppose the slowly varing signals (the voice, music, etc.) are $\sin t$ and $\sin 2t$. The chart below shows, for three stations, what they each want to send in terms of $\sin t$ and $\sin 2t$ and what the carrier wave is for each station.

Station Name	sending function	carrier wave function	total signal sent
S_1	$3\sin t - 2\sin 2t$	$\sin 8t$	$(3\sin t - 2\sin 2t)\sin 8t$
S_2	$5\sin t + 6\sin 2t$	$\sin 16t$	$(5\sin t + 6\sin 2t)\sin 16t$
S_3	$4\sin t + 7\sin 2t$	$\sin 24t$	$(4\sin t + 7\sin 2t)\sin 24t$

The signals are all broadcast, which in mathematical terms means the functions are all super imposed on each other. So the signal reaching our radio is

$$f(t) = (3\sin t - 2\sin 2t)\sin 8t + (5\sin t + 6\sin 2t)\sin 16t + (4\sin t + 7\sin 2t)\sin 24t$$

Plot each station's signal and its sending function on its own graph, and also plot the signal our radio is receiving on its own graph. (So 4 graphs total here)
So how do we 'tune' to say, station S_2 and recover the signal? First, use the work you did in problem 4 above to write $f(t)$ in the following form: (you will need to determine the values of A, B, C, D, E, and F)

$$f(t) = A(\cos 7t - \cos 9t) - B(\cos 6t - \cos 10t) + C(\cos 15t - \cos 17t) + D(\cos 14t - \cos 18t) + E(\cos 23t - \cos 25t) + F(\cos 22t - \cos 26t)$$

Problem 7 (100 points) Now, use the work you did in problem 4 above to show the following, and then multiply by -4π to recover the coefficients of $\sin t$ and $\sin 2t$ for station S_2.

$$\int_0^\pi f(t)\cos 17t\,dt = \frac{-5}{2}\frac{\pi}{2} \qquad \int_0^\pi f(t)\cos 18t\,dt = \frac{-6}{2}\frac{\pi}{2}$$

So now you should have the basic idea of how Fourier series are used in radio (and indeed many, many other places). Your work in problem 1 should have indicated how almost any function could be approximated by sines and cosines, and then your work in problem 6 gives an indication of how we can recover one signal from many.