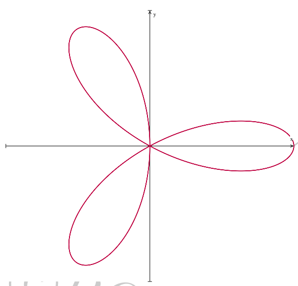
Math 2414 Polar Examples

Problem 2 (20 points) Eliminate the parameter to obtain an equation in x and y. Describe the curve and indicate the positive orientation, and give the starting and ending point of the curve.

$$x = \cos t \qquad y = \sin^2 t \qquad 0 \le t \le \pi$$

One way is $t = \cos^{-1} x$, so $y = \sin^2(\cos^{-1} x)$, but that is clunky. The other was is to recognize that $\sin^2 t + \cos^2 t = 1$ means that $y + x^2 = 1 \to y = 1 - x^2$ So on $[0, \pi]$ this is part of the parabola that opens downward with vertex at $(0, 1)$. The orientation is counterclockwise, from $(1, 0)$ through the vertex to $(-1, 0)$.

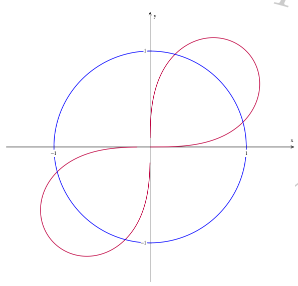
Problem 3 (30 points) Find the area of the following regions: The region inside one leaf of $r = \cos 3\theta$



First we need to know when $0 = \cos 3\theta$. This happens the first time at $\pi/6$. So we need the following, which we will then double:

$$\frac{1}{2}\int_0^{\pi/6} (\cos(3\theta))^2\, d\theta = \frac{1}{2}\int_0^{\pi/6} \frac{\cos(6\theta) + 1}{2} d\theta = \frac{\sin 6\theta}{24} + \frac{\theta}{4}\Big|_0^{\pi/6} = \frac{\pi}{24} \to \frac{\pi}{12}$$

The region inside the lemniscate $r^2 = 2\sin 2\theta$ and outside $r = 1$

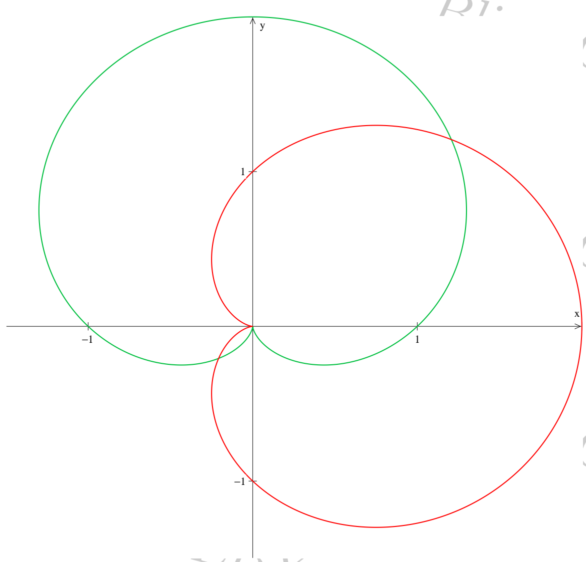


Again, we start by finding where the curves intersect. $2\sin 2\theta = 1^2$ gives $\pi/12$ and $5\pi/12$ as the first two places. This will find the top right *bubble*, so we will double it to find the total area. Because of the way the functions are defined, we do NOT get a sin or cos squared term

$$\frac{1}{2}\int_{\pi/12}^{5\pi/12} (2\sin 2\theta - 1)\, d\theta = \int_{\pi/12}^{5\pi/12} (\sin 2\theta - 1/2)\, d\theta = \frac{-1}{2}\cos(2\theta) - \frac{\theta}{2}\Big|_{\pi/12}^{5\pi/12} = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

Problem 4 (30 points) Find the intersection points and then find the area that lies within both of the following pairs of curves.

$$r = 1 + \sin\theta \qquad r = 1 + \cos\theta$$



So we want when $1 + \sin\theta = 1 + \cos\theta$ This happens at $\theta = \pi/4$ and $\theta = 5\pi/4$ But if you look at the graph, we have to be careful.
On $[, \pi/4]$, $1 + \sin\theta$ is inside $1 + \cos\theta$
then on $\pi/4, 5\pi/4]$ it reverses
then on $[5\pi/4, 2\pi]$, $1 + \sin\theta$ is again inside. So we need the following:

$$\frac{1}{2}\int_0^{\pi/4} (1 + \sin(\theta))^2\, d\theta + \frac{1}{2}\int_{\pi/4}^{5\pi/4} (1 + \cos(\theta))^2\, d\theta + \frac{1}{2}\int_{5\pi/4}^{2\pi} (1 + \sin(\theta))^2\, d\theta$$