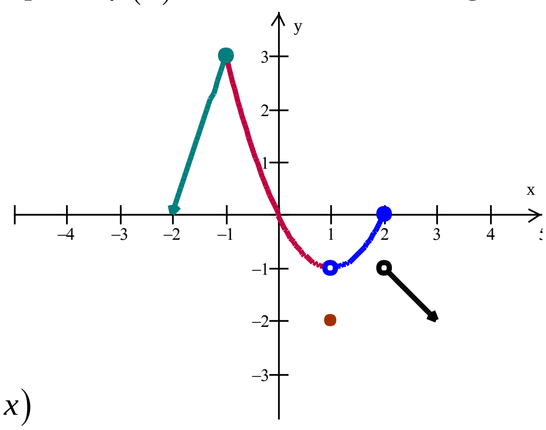
Math 2413 – Final Exam Review

1. Evaluate, giving exact values when possible.
   a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
   b. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
   c. $\cos\left(\sin^{-1}\frac{y}{2}\right)$

2. Evaluate the expression.
   a. $2^{\log_2 3-\log_2 5}$
   b. $e^{5\ln 2}$

3. Solve for x.
   a. $4^{x^2+x}=16$
   b. $e^{2x+3}=1$

4. Use the given graph of $f(x)$ to answer the following:



   a. $\lim_{x\to -1^-} f(x)$
   b. $\lim_{x\to -1^+} f(x)$
   c. $\lim_{x\to -1} f(x)$
   d. $\lim_{x\to 2^-} f(x)$
   e. $\lim_{x\to 2^+} f(x)$
   f. $\lim_{x\to 2} f(x)$
   g. List all *x*-coordinates where f is not continuous and explain your answer mathematically.
   h. List all *x*-coordinates where f is not differentiable and explain why.

5. Find the average rate of change of the function on the given interval.

   a. $g(\theta) = 8 - \tan\theta$, $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

   b. $f(x) = \cot x$, $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

6. Evaluate each limit, if it exists.

   a. $\lim\limits_{h\to 0} \dfrac{(x+h)^3 - x^3}{h}$ This is a conceptual problem.

   b. $\lim\limits_{h\to 0} \dfrac{e^{x+h} - e^x}{h}$ This is a conceptual problem.

   c. $\lim\limits_{h\to 0} \dfrac{\tan^{-1}(x+h) - \tan^{-1}(x)}{h}$ This is a conceptual problem.

   d. $\lim\limits_{x\to 2} x^3 - 5x^2 + 4$

   e. $\lim\limits_{x\to -1} \dfrac{x^2 - 2x - 3}{x+1}$

   f. $\lim\limits_{x\to 2^+} \dfrac{3+x}{2-x}$

   g. $\lim\limits_{x\to 3^-} \dfrac{|x-3|}{x-3}$

   h. $\lim\limits_{x\to \infty} \dfrac{9x^5 + 50x^2 + 800}{x^5 - 1000}$

   i. $\lim\limits_{h\to 0} \dfrac{(4+h)^2 - 16}{h}$

   j. $\lim\limits_{h\to 0} \dfrac{3 - \sqrt{9+h}}{h}$

   k. $\lim\limits_{x\to 1} \dfrac{\frac{1}{x} - 1}{x-1}$

   l. $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\ln(\csc x)}{\left(x - \frac{\pi}{2}\right)^2}$

   m. $\lim\limits_{x\to 0^+} \dfrac{\ln(e^x - 1)}{\ln x}$

   n. $\lim\limits_{x\to 0^+} \left(1 + \dfrac{1}{x}\right)^x$

   o. $\lim\limits_{x\to \infty} \left(1 - \dfrac{2}{x}\right)^x$

p. $\lim_{x\to\infty}\left(x^{-5}\ln x\right)$

q. $\lim_{\theta\to-\infty}\frac{\cos\theta}{3\theta}$

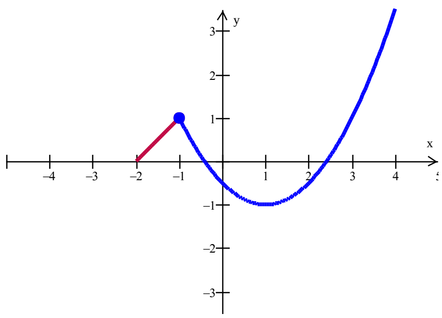
r. $\lim_{x\to 1}\frac{x^3-1}{x^2-1}$

7. Find constants a and/or b so that $f(x)$ will be continuous for all values of x.

a. $f(x)=\begin{cases} ax+3 & \text{if } x>5 \\ 8 & \text{if } x=5 \\ x^2+bx+1 & \text{if } x<5 \end{cases}$

b. $f(x)=\begin{cases} b, & x\ge 1 \\ 8b^2x^2+b^2x, & x<1 \end{cases}$

8. Given the graph of f, sketch the graph of f'.



9. Determine if each of the given functions is differentiable and continuous everywhere it is defined.

a. $f(x)=\begin{cases} 4 & \text{if } x<2 \\ 2x & \text{if } x\ge 2 \end{cases}$

b. $f(x)=\begin{cases} 2x+1 & \text{if } x<0 \\ 3x+1 & \text{if } x\ge 0 \end{cases}$

c. $f(x)=\begin{cases} x^2 & \text{if } x<0 \\ x^3 & \text{if } x\ge 0 \end{cases}$

d. $f(x)=\begin{cases} x & \text{if } 0<x<1 \\ 2-x & \text{if } 1\le x<2 \end{cases}$

e. $f(x)=\begin{cases} x & \text{if } -1<x<0 \\ \tan(x) & \text{if } 0\le x<\frac{\pi}{4} \end{cases}$

10. Consider the function $f(x) = 2x^2 - x + 3$.

a. Use the definition of derivative to determine $f'(x)$.

b. Find an equation of the line tangent to the graph of $f(x)$ at $x = 1$.

11. Differentiate the following. Simplify your answers completely.

a. $f(x) = x^2 - \dfrac{1}{x^2} + \sqrt[3]{x^2} + \pi^2$

b. $f(x) = \dfrac{3x+5}{x+9}$

c. $f(x) = (3x+5)^2 (2x^3 + x^2)^3$

d. $f(x) = \sec(4x)$

e. $f(x) = x^2 \ln(3x-1)$

f. $f(x) = 3e^{5x-4}$

g. $f(x) = \sin^{-1}(x^2)$

h. $f(x) = \dfrac{\sec x}{\tan x}$

i. $f(x) = \dfrac{x}{\cos x} - \dfrac{x}{\sin x}$

j. $f(x) = \tan^2(3x)\sec(x)$

k. $f(x) = \log_3(2x^2 + 1) + 6^{(3x^3)}$

12. Find $f^{(4)}(x)$ of the following.

a. $f(x) = x^5 - 5x^3 + x + 12$

b. $f(x) = \dfrac{x+1}{x-2}$

c. $f(x) = -\sin x$

d. $f(x) = e^{5x}$

e. $f(x) = \cos(4x)$

13. Suppose $f(x)$ and $g(x)$ are defined so that the following table of values holds true. Evaluate the derivative of $h(x)$ at the indicated *x*-value.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	–3	1/2
1	3	5	1/2	–4

a. $h(x)=5f(x)-4g(x); \quad x=1$

b. $h(x)=f^2(x)g^3(x); \quad x=0$

c. $h(x)=g\left(x^2+f(x)\right); \quad x=0$

d. $h(x)=\left(2x+g(x)\right)^3; \quad x=1$

14. Calculate the instantaneous rate of change of the given function at the specified value x_0.

a. $f(x)=\sin x\cos x,\ x_0=\dfrac{\pi}{2}$

b. $f(x)=\sqrt{x}\left(2-x^3\right),\ x_0=1$

15. Find the slope of the line tangent to $f(x)=\tan x$ at $x=\dfrac{\pi}{4}$.

16. Find the velocity and acceleration at time t when $s(t)=t^4-4t^3+8t$.

17. Find the x-coordinate of each point where the graph of $f(x)=\dfrac{(x-1)^2}{(x+2)^3}$ has a horizontal tangent line.

18. Find the equation for the line tangent to $y=x^2+\sin y$ at $\left(\sqrt{\pi},\pi\right)$.

19. Find $\dfrac{dy}{dx}$ for each of the following.

a. $x^2+3xy+y^2=15$

b. $\ln(xy)=e^{2x}$

c. $y^2+3=x\sec y$

d. $y=x^x$

e. $y=\sin^{-1}(2x+1)$

f. $y=x^2\tan^{-1}(3x)$

20. An airplane is flying at 150 ft/s at an altitude of 2000 ft in a direction that will take it directly over an observer at ground level. Find the rate of change of the angle of elevation between the observer and the plane when the plane is directly over a point on the ground that is 2000 ft from the observer.

21. A 50-ft ladder is placed against a large building. The base of the ladder is resting on an oil spill, and it slips away from the building at a rate of 3 ft per minute. Find the rate of change of the height of the top of the ladder above the ground at the instant when the base of the ladder is 30 ft from the base of the building.

22. A sand storage tank used by the highway department for winter storms is leaking. As the sand leaks out, it forms a conical pile. The radius of the base of the pile increases at the rate of 0.75 in. per minute. The height of the pile is always twice the radius of the base. Find the rate at which the volume of the pile increasing at the instant the radius of the base is 6 in. Leave your answer in terms of π.

23. Find all absolute extrema for f on the given interval.

    a. $f(x) = \sin x \cos x,\ x \in \left[0, \frac{\pi}{2}\right]$

    b. $f(x) = x^5 - x^4,\ [-1,1]$

    c. $f(x) = |x-1|,\ [-2,2]$

24. Find all interval(s) on which f is increasing or decreasing.

    a. $f(x) = 9x^{2/3} - 3x$

    b. $f(x) = x^3 + 3x^2 - 24x + 2$

    c. $f(x) = \dfrac{1}{x+2}$

25. Find all points of inflection and state all intervals of concavity for the following functions.

    a. $f(x) = \frac{1}{5}x^6 - 2x^4$

    b. $f(x) = x\left(x + \frac{1}{x}\right)$

26. The function $f(x) = x^3 + ax^2 + bx + 7$ has a relative extrema at $x = 1$ and $x = -3$.

    a. Find a and b.

    b. Use the Second Derivative Test for Relative Extrema to classify each extremum as a relative maximum or a relative minimum.

    c. Determine the relative extrema.

27. Let $f(x)$ be a polynomial function that has critical numbers -3, -1 and 2. Furthermore, assume $f''(x) = 3x^2 + 4x - 5$. For which critical numbers does the Second Derivative Test indicate the function f has a relative minimum?

28. Sketch the graph of a function with the following properties.

$f'(x) > 0$ on $(-\infty, -5) \cup (1, \infty)$

$f'(x) < 0$ on $(-5, 1)$

$f'(-5) = 0$

$f'(1) = 0$

$f(-5) = 4$

$f(1) = -1$

29. Sketch the graph of a function with the following properties.

a. Continuous and differentiable on $(3, \infty)$

b. $g(4) = 0$

c. $\lim_{x \to 3^+} g(x) = -\infty$

d. $g'(x) > 0$ on $(3, \infty)$

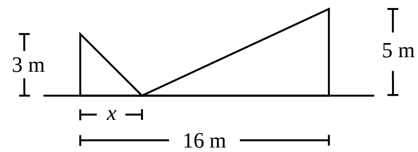
e. $g''(x) < 0$ on $(3, \infty)$

30. Consider the points *A*, *B*, and *C*. A company wishes to run a utility cable from point *A* to point *B*. Point *A* lies 9 miles directly east of point *C* along a straight shoreline. Point *B* is located on an island 6 miles directly north of point *C*. It costs $400 per mile to run the cable on land and $500 per mile underwater. Assume that the cable starts at point *A* and runs along the shoreline, then angles and runs underwater to the island. Find the distance from point *A* at which the cable line should begin to run underwater in order to yield the minimum total cost.

31. An open box is to be made from a square piece of cardboard measuring 15 inches on a side by cutting a square from each corner and folding up the sides. Find the dimensions for which the volume of the resulting box is maximized.

32. A man is floating in a rowboat 1 mile from the (straight) shoreline of a large lake. A town is located on the shoreline 1 mile from the point on the shoreline closest to the man. He intends to row in a straight line to some point *P* on the shoreline and then walk the remaining distance to the town. To what point should he row in order to reach his destination in the least amount of time if he can walk 5 mph and row 3 mph? State your answer as a distance from the town.

33. Two posts, one 3 meters high and one 5 meters high, stand 16 meters apart. They are tied to a single stake by two wires running from ground level to the top of each post. How far from the 3-m post should the stake be placed to use the least amount of wire?



34. Evaluate the following.

a. $\int \csc\theta \cot\theta \, d\theta$

b. $\int \left(-18t^2 + 5t - \frac{4}{t}\right) dt$

c. $\int \frac{dx}{\sqrt{1-x^2}}$

d. $\int \frac{x\,dx}{\sqrt{1-x^2}}$

e. $\int \left(x^2-2\right) e^{x^3-6x}\, dx$

f. $\int \frac{x^2}{x^3+1}\, dx$

g. $\int \sec^2(2t)\tan(2t)\, dt$

h. $\int \frac{\sin^{-1}(7x)}{\sqrt{1-49x^2}}\, dx$

i. $\int \frac{1}{9x^2+4}\, dx$

j. $\int_{-\pi/3}^{\pi/3} \sec^2\theta \, d\theta$

k. $\int_4^{16} \frac{1}{\sqrt{x}}\, dx$

l. $\int_0^1 \frac{4}{1+\theta^2}\, d\theta$

m. $\int_0^1 \left(5x^3 + ax\right) dx$

35. Evaluate the sums using known summation formulas.

a. $\sum_{k=1}^{9} -\frac{k}{5}$

b. $\sum_{k=1}^{14} \left(3-k^2\right)$

c. $\sum_{k=1}^{5} k\left(6k^2+5\right)$

36. Use the definition of the definite integral to evaluate $\int_0^4 (4x+1)\,dx$ using right Riemann sums.

37. Simplify the following expressions showing all steps and using correct mathematical notation.

a. $\frac{d}{dx}\left(\int_0^{3x^2} \frac{t^2-1}{\sqrt{t+1}}\,dt\right)$

b. $\frac{d}{dx}\left(\int_x^{1} t^3 \cos^2 t\,dt\right)$

c. $\frac{d}{dx}\left(\int_0^{x} \left(\sec^3 t \tan^2 t\right)dt\right)$

38. Find the area of the region under each of the given curves for the given interval.

a. $y = x^2 + x - 6$ on $[-1,4]$

b. $y = \cos\theta$ on $\left[\frac{\pi}{6}, \frac{2\pi}{3}\right]$

c. $y = x^4 - 4x^2$ on $[-2,0]$

39. Assume $a(t)$ represents the acceleration of an object at time t, $v(t)$ represents the velocity of the object at time t, and $s(t)$ represents the position of the object at time t. If $a(t) = 2 + \sin t$, $v(0) = 4$ m/s, and $s(0) = 5$ m, find $s(t)$.

Topics

Limits and Continuity

- Be able to find limits by simplifying the expressions. Consider reducing rational expressions, multiplying by conjugates, or by simplifying an absolute value expression.
- Be able to find limits at infinity by dividing by the highest power of x in the denominator. Know when this process will work better than using L'Hôpital's Rule.
- Know L'Hôpital's Rule and be able to use it to find limits of indeterminate forms. Know when L'Hôpital's Rule applies and when it doesn't. Be able to recognize all of the indeterminate forms. Be able to use the natural log function to work with exponential functions.
- Be able to use the definition of the derivative to evaluate a limit.
- Know the definition of continuity. Be able to decide if a function is continuous and to be able to find constants so that a function is continuous.

Derivatives

- Know the definition of a derivative. Be able to find the derivative using only the definition. Be able to find a limit by recognizing it as a derivative.
- Know all of the derivative short-cut rules.
- Be able to use implicit differentiation. Remember that the chain, product, and quotient rules still apply.
- Be able to find a derivative by using the Second Fundamental Theorem of Calculus.

Applications of the Derivative

- Be able to find the slope of a tangent or a normal line.
- Know the relationship between a position (distance) function, the velocity and the acceleration.
- Be able to solve related rates problems. Look for the question "how fast" or "what rate".
- Be able to find absolute and local extrema. Be able to decide if your answer should be a y-value or if it should be a point. Be able to use both the first and second derivative tests.
- Know what information derivatives give about the shape of a function. Be able to find intervals of increase/decrease, intervals of concavity, vertical tangents and cusps.
- Be able to solve applied optimization problems.

Integrals

- Know all of the antiderivative rules.
- Be able to evaluate definite and indefinite integrals.
- Given an acceleration or velocity function and some initial values, be able to find the position function.
- Be able to use integration by substitution to evaluate definite and indefinite integrals.
- Know the Fundamental Theorem of Calculus -- both parts. Be able to evaluate a definite integral by using the First Fundamental Theorem of Calculus. Be able to find the derivative of a function defined as a definite integral by using the Second Fundamental Theorem of Calculus.